Deformation Theory

Talk 7

Talk 7. Cohen-Macaulay schemes in codimension 2.

Goal. In talk 4 we proved that defs. of non-singular affine schemes are trivial, so it is natural to study deformations of closed subschemes of a non-singular scheme. (embedded deformations). We will prove that CM-schemes in codim. 2 comes with a particular free resolution that is liftable to any deformation. This will prove that deformations are locally unobstructed in this class of schemes.

§1. Cohen-Macaulayness

Let A be a ring, $M \in A$-<u>mod</u>. A reg. sequence for M is a set of elements $x_1, \dots, x_r \in A$ s.t. $\forall i$, x_i is not a zero-divisor in $M / \langle x_1, \dots, x_{i-1} \rangle M$.

Interpretation of depth. Geometrically, depth measures how many times we can "cut down by hypersurfaces", and in general for any local ring A it is verified

$$\operatorname{depth} A \leq \dim A$$

Def. Let $(A, \mathfrak{m})$ be a local ring, $M \in A\text{-}\underline{\mathrm{mod}}$. The depth of M is the maximum length of a reg. sequence. We say that A is a Cohen-Macaulay (CM) ring if $\operatorname{depth} A = \dim A$.

Prop. Let $(A, \mathfrak{m})$ be a local noetherian ring. Then

(1) A regular $\Rightarrow$ A CM

(2) A CM $\Rightarrow$ $A_{\mathfrak{p}}$ CM $\forall\ \mathfrak{p} \in \operatorname{Spec} A$.

Def. The projective dim. of $M \in A\text{-}\underline{\mathrm{mod}}$ if the minimum length of a projective resolution of M, i.e., of an exact sequence

$$\cdots \to L_1 \to L_0 \to M \to 0$$

with L_i projective.

Thm (Auslander-Buchsbaum). If M is a f.g. $(A, \mathfrak{m})$-module, then

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A$$

Def. A scheme X is called CM if $\mathcal{O}_{X,x}$ is CM $\forall\ x \in X$.

Example. A l.c.i on a non-singular variety is CM.

§2. CM schemes of codim 2.

First, we will prove a structure theorem for CM quotients A/I of codim. 2.

Theorem (Hilbert-Burch). Let A be a regular local ring of $\dim A = n$, $B = A/I$ with $\dim B = n-2$ and s.t. B is CM. There exists an exact sequence

$$0 \to A^r \xrightarrow{\varphi} A^{r+1} \to A \to B \to 0$$

where φ is given by a matrix s.t. its maximal minors are a minimal set of generators of I.

Proof. By Auslander-Buchsbaum thm.

$$\operatorname{depth} B + \operatorname{pd} B = \operatorname{depth} A$$

$\rightsquigarrow \operatorname{pd} B = 2$. If we take a minimal set of generators $a_1, \dots, a_{r+1}$ for I, we obtain an exact sequence

$$0 \to A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{\alpha} A \to B \to 0$$

where α is the map given by $e_i \mapsto a_i$ and φ is given by an $(r+1) \times r$ matrix.

rows

$$\varphi = \begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \end{pmatrix}, \quad \text{define } f : e_i \longrightarrow (-1)^i \begin{vmatrix} a_1 \\ \vdots \\ \cancel{a_i} \\ \vdots \\ a_{r+1} \end{vmatrix} = M_i$$

Notice that

$$(f \circ \varphi)(e_j) = f(c_j) = \sum_{i=1}^{r+1} (-1)^i a_{ij} M_i$$

computes the determinant of the $(r+1) \times (r+1)$ matrix that results from adding the column c_j to the matrix $\varphi \rightsquigarrow f \circ \varphi = 0$, and we have a complex

$$A^r \xrightarrow{\varphi} A^{r+1} \xrightarrow{f} A \qquad (*)$$

Now, we can show f is proportional to α. Notice that since φ is injective, it has rank r so $f \neq 0$. Consider $K = \mathrm{Frac}(A)$ and tensor to obtain

$$K^r \xrightarrow{\varphi_K} K^{r+1} \xrightarrow{f_K} K$$

and notice that comparison of ranks shows homology of $(*)$ has rank $0 \rightsquigarrow$ is just torsion elements. Now, since

$$\ker f / \operatorname{Im} \varphi \hookrightarrow A^{r+1} / \operatorname{Im} \varphi = \operatorname{coker} \varphi$$

and $\operatorname{coker} \varphi \simeq A^{r+1} / \ker \alpha \simeq \operatorname{Im} \alpha \simeq I \subseteq A$ is torsion-free $\rightsquigarrow (*)$ is exact.

This implies that $I = J = \langle f_1, \dots, f_{r+1} \rangle$.

Consider a point $\mathfrak{p} \in \operatorname{Spec} A$ of codim 1, so $\mathfrak{p} \in \operatorname{Spec} B$ and then

$$0 \to A_{\mathfrak{p}}^{r} \xrightarrow{\varphi_{\mathfrak{p}}} A_{\mathfrak{p}}^{r+1} \xrightarrow{\alpha} A_{\mathfrak{p}} \to B_{\mathfrak{p}} = 0$$

and since $A_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-mod, this sequence is split, i.e., $\operatorname{Im} \varphi_{\mathfrak{p}}$ is a direct summand of $A_{\mathfrak{p}}^{r+1}$ and there exists $\chi : A_{\mathfrak{p}}^{r+1} \to A_{\mathfrak{p}}^{r}$ s.t. $\chi \circ \varphi_{\mathfrak{p}} = \mathrm{id}$, i.e., there exists a $r \times (r+1)$ matrix χ s.t. $\det(\chi \circ \varphi_{\mathfrak{p}}) \neq 0 \rightsquigarrow$ some f_i is a unit in $A_{\mathfrak{p}}$, and then A/J is supported in codimmsion ≥ 2. Since $I \simeq J$ as A-mods, $\exists\ a, b \in A \setminus \{0\}$ s.t. $aI = bJ$.

On the other hand, since A is a regular ring $A = \bigcap_{ht(\mathfrak{p})=1} A_{\mathfrak{p}}$ and $I_{\mathfrak{p}} = A_{\mathfrak{p}}$, $J_{\mathfrak{p}} = A_{\mathfrak{p}}$ $\forall\ \mathfrak{p} \in \operatorname{Spec} A$ with $ht(\mathfrak{p}) = 1$, so, $a_{\mathfrak{p}} A_{\mathfrak{p}} = b_{\mathfrak{p}} A_{\mathfrak{p}}$ and this implies $a, b \in A^{\times} \rightsquigarrow I = J$ ∎

In the geometric setting we obtain.

Corollary. Let X be a smooth variety $/k$, $Y \subseteq X$ a CM closed subscheme of $\operatorname{codim}_X Y = 2$. Then $\forall\, y \in Y$, $\exists$ an open aff. nbd. $y \in U \subseteq X$ and φ a matrix of reg. functions in $\mathcal{O}_X(U)$ s.t. there is an exact sequence

$$0 \to \mathcal{O}_U^r \xrightarrow{\varphi} \mathcal{O}_U^{r+1} \xrightarrow{f} \mathcal{O}_U \to \mathcal{O}_{Y \cap U} \to 0$$

and the maximal minors of φ generates $\mathcal{I}_X(Y \cap U)$.

Next time. We will prove that local deformations of Y are given by liftings of the matrix φ.