



## § 14. Plane curve singularities

**Idea of local moduli.** Try to describe all possible local deformations of a plane curve singularity.

**Goal.** To find  $X \rightarrow \text{Spec } R$  deformation being

- "complete" (versal): any local def. can be obtained by base extension from this one.
- "minimal" (miniversal): it is the smallest possible.
- "universal": any other def. is obtained by unique base extension from this one.

**Example.** Consider the node  $X_0 = (xy = 0) \subseteq \mathbb{A}^2 = \text{Spec } k[x, y]$

and  $X = (xy - t = 0) \subseteq \mathbb{A}^3 = \text{Spec } k[x, y, t]$

$\downarrow$

$T = \text{Spec } k[t].$

$t \neq 0 \rightsquigarrow X_t$  hyperbola

$t = 0 \rightsquigarrow X_0$  node

Let  $X'/S$  be any deformation of  $X_0$  over  $S = \text{Spec } \hat{R}$  with  $\hat{R}$  a complete local ring.

For simplicity we assume

-  $\hat{R} = k[[s]]$

-  $X'$  is defined by one equation  $g(x, y, s) = 0$ ,  
 $g \in k[[s]][x, y]$  with  $g(x, y, 0) = xy$ .

(since  $\mathbb{A}^2 \times \text{Spec } k[[s]] = \text{Spec } k[[s]][x, y] =: \mathbb{A}_{\hat{R}}^2$ )

**Aim.** There exists  $S \rightarrow T$  (i.e.,  $\mathbb{K}[t] \xrightarrow{Y} \mathbb{K}[s]$ )  
 given by  $\varphi(t) = T(s)$ ,  $T(0) = 0$  s.t.  $X' \cong X \times_T S$ .

The last condition means the following:

- Pullback of universal family will be given by the equation

$$xy - T(s) = 0 \text{ in } \hat{A}_{\mathbb{R}}^2,$$

thus we need an isomorphism

$$\hat{R}[x, y] / \langle xy - T(s) \rangle \cong \hat{R}[x, y] / \langle g \rangle$$

Then, it is enough to construct

$$X(x, y, s), Y(x, y, s), U(x, y, s) \in \hat{R}[x, y]$$

s.t.  $\hookrightarrow$  change of variables

- $X(x, y, 0) = x$
- $Y(x, y, 0) = y$
- $U(x, y, s)$  is a unit and  $U(x, y, 0) = 1$ .
- $U(XY - T) = g(x, y, s)$  (\*)

**Construction.** We have given  $T(0) = 0$ ,  $X(0) = x$ ,  
 $Y(0) = y$ ,  $U(0) = 1$ ,  $g = xy + \sum_{i \geq 1} g_i s^i$ ,  $g_i \in \mathbb{K}[x, y]$ .

$$\text{Let } T = \sum a_i s^i, \quad X = x + \sum b_i s^i, \quad Y = y + \sum c_i s^i$$

$$U = 1 + \sum u_i s^i, \quad a_i \in \mathbb{K}, \quad b_i, c_i \in \mathbb{K}[x, y].$$

Substituting (\*) we see

$$U(XY - T) = g(x, y, s)$$

$$(1 + u_1 s + \dots)((x + b_1 s + \dots)(y + c_1 s + \dots) - a_1 s - \dots) = xy + g_1 s + \dots$$

$$xy + b_1 sy + c_1 sx - a_1 s + u_1 sxy + \dots = xy + g_1 s + \dots$$

and deg 1 part (in  $s$ ) of (\*) is given by

$$b_1 y + c_1 x - a_1 + u_1 xy = g_1 \quad (*),_1$$

We can choose  $a_1, b_1, c_1, u_1$  s.t.  $(*)_1$  holds.

Then, by induction we can suppose  $a_i, b_i, c_i, u_i$  were already constructed for  $i < n$ , and then deg.  $n$  part of (\*) gives

$$h(x, y) + x c_n + y b_n - a_n + xy u_n = g_n$$

and we can solve this.

**Problem.** We constructed  $X, Y, U$  in the ring  $K[x, y][[s]]$  (bigger than  $K[[s]][x, y]$ ).

**Recall.** Let  $X$  be a noetherian scheme,  $Y \subseteq X$  closed subscheme. The formal completion of  $X$  along  $Y$ , denoted  $(\hat{X}, \hat{\mathcal{O}}_{\hat{X}})$ , is the ringed space given by:

- topological space:  $Y$
- sheaf of rings:  $\hat{\mathcal{O}}_{\hat{X}} = \varprojlim \mathcal{O}_X / \mathcal{I}_Y^n$ .

**Conclusion** The formal completions of  $X'$  and  $X \times_T S$  along the closed fiber at  $s=0$  are isomorphic. Thus, the property we expected just hold at this level.

**Remarks.** (1) Everything works in the same way if we also consider power series in  $x, y$ .

(2) Linear coeff.  $a_1$  is uniquely determined, and notice that differential of  $\gamma: S \rightarrow T$  is given by

$$d_0\gamma: \mathfrak{m}_S/\mathfrak{m}_S^2 \cong S/S^2 \rightarrow \mathfrak{m}_{T,0}/\mathfrak{m}_{T,0}^2 \cong t/t^2, \quad s \mapsto a_1 s$$

This is interpreted as a minimality property (we are using the minimum number of parameters we need to describe def. of  $X_0$ ), and we say  $T$  is the minimal def. space of  $X_0$ .

(3)  $a_i$  for  $i > 1$  are not uniquely determined, so  $S \rightarrow T$  is not unique (def. is not universal).

(4) In the previous example we could have taken  $u_n = 0 \quad \forall n \geq 1$ , but it is included because in the general case  $f(x, y) = 0$  might be not homogeneous.

**General case.** Consider an isolated plane curve sing.

given by  $f(x, y) = 0$  that we assume is at the origin,

so the ideal  $J = \langle f, f_x, f_y \rangle$  is primary to

$$\mathfrak{m} = \langle x, y \rangle.$$

Recall (talk 4). Let  $R = k[x, y]$ ,  $I = \langle f \rangle$ ,  $B = R/I$

$\rightarrow$  defs. of  $B$  over dual numbers are parametrized by  $T^1(B/k, B)$ .

Here we can compute explicitly. Recall that

$$T^1(B/k, M) = \text{coker}(\text{Hom}(\Omega_{A/k}, M) \xrightarrow{f^*} \text{Hom}(I/I^2, M))$$

$$f: I/I^2 \rightarrow \Omega_{A/k} \otimes_A B, \quad I/I^2 = \langle \bar{f} \rangle$$

$$\bar{f} \mapsto db \otimes 1 \quad d: \bar{f} \mapsto f_x dx + f_y dy$$

$$\text{Hom}(\Omega_{A/k}, M) \xrightarrow{f^*} \text{Hom}(I/I^2, M)$$

$\psi$

$\psi \circ f$

$$\psi(dx) = m_1$$

$$\psi(dy) = m_2$$

$$\psi \circ f(\bar{f}) = \psi(f_x dx + f_y dy) = f_x m_1 + f_y m_2$$

$$\leadsto T^1(B/k, M) \simeq M / \langle f_x, f_y \rangle M$$

$$\leadsto T^1(B/k, B) \simeq B / \langle f_x, f_y \rangle = R/J.$$

Take  $g_1, \dots, g_r \in R/J$  whose image in  $R/J$  are a basis as vector space. Consider the ring

$T = \text{Spec } k[T_1, \dots, T_r]$  and the deformation  $X/T$  given by

$$F(x, y, t) = f(x, y) + \sum_{i=1}^r t_i g_i(x, y) = 0$$

**Theorem.** Let  $f(x,y) = 0$  an isolated plane curve sing. and  $X/T$  the constructed deformation. Then  $X/T$  is the miniversal def. space, meaning

- (1)  $\forall X'/S$  def. with  $S = \text{Spec } \hat{R}$  complete local ring,  $\exists \psi: S \rightarrow T$  s.t.  $X'$  and  $X \times_T S$  become isomorphic after completing along the closed fiber over zero.
- (2) although  $\psi$  is not unique, the induced map on Zariski tangent spaces of  $S$  and  $T$  is uniquely determined.

**Example.** Consider the cusp  $f(x,y) = y^2 - x^3$   
 $\rightarrow f_x = -3x^2, f_y = 2y$  (assume  $\text{char } k \neq 2, 3$ ).  
 $\Rightarrow k[x,y]/\langle f, f_x, f_y \rangle$  is generated by  $\{1, x\}$  over  $k$ .

Let's consider the miniversal deformation

$$F(x,y;t,u) = y^2 - x^3 + t + ux = 0$$

For gen.  $t, u$  the system

$$\begin{cases} F = 0 \\ F_x = -3x^2 + u \\ F_y = 2y \end{cases} \text{ has no solution} \quad (\text{the curve defined by } F \text{ is smooth})$$

$$\text{Now, } \begin{cases} F_y = 0 \rightsquigarrow y = 0 \\ F_x = 0 \rightsquigarrow u = 3x^2 \end{cases} \rightsquigarrow \begin{cases} t = x^3 - ux \\ u = 3x^2 \end{cases} \Rightarrow \begin{cases} t = -2x^3 \\ u = 3x^2 \end{cases}$$

$\rightsquigarrow$  singular fibers happen over  $\Delta = (27t^2 - 4u^3 = 0)$ .