Deformation Theory Group

Talk 5

Obstruction to deformations

Recall. Consider X_0 sch./k, C an Artin local k-algebra. A def. of X_0 over C is a scheme X flat$/C$ with a closed immersion $X_0 \hookrightarrow X$ s.t. $X \times_C k \simeq X_0$.

Def. For a map $C' \twoheadrightarrow C$ of Artin local k-algebras, we say that

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} C' \end{tikzcd} \quad \text{extends} \quad \begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

if there exists a comm. diagram

$$\begin{tikzcd} X_0 \arrow[rr] \arrow[dr, hook] \arrow[dd] & & X' \arrow[dd] \\ & X \arrow[ur, hook] \arrow[dd] & \\ \operatorname{Spec} k \arrow[rr] \arrow[dr, hook] & & \operatorname{Spec} C' \\ & \operatorname{Spec} C \arrow[ur, hook] & \end{tikzcd}$$

s.t. $X' \times_{C'} C \simeq X$.

Two extensions are equiv. if $\exists$ an isom. $/C'$ s.t. compatible with the immersions of X_0.

Question. Given a def.

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

and $C' \twoheadrightarrow C$

(1) Is there exists an extension?

(2) How many?

Affine case. $X_0 = \operatorname{Spec} B_0$, $B_0 = k[x_1, \dots, x_n]/I_0$, $X = \operatorname{Spec} B$, $B = C[x_1, \dots, x_n]/I$, $X' = \operatorname{Spec} B'$, $B' = C'[x_1, \dots, x_n]/I'$.

Consider $0 \to J \to C' \to C \to 0$ (*) with $J^2 = 0$. Then M' C'-mod is flat $/C'$ $\iff$ $M = M' \otimes_{C'} C$ flat$/C$ and $M \otimes_C J \hookrightarrow M'$ (talk 2)

Let $R = C[x_1, \dots, x_n]$, $R' = C'[x_1, \dots, x_n]$, $R_0 = k[x_1, \dots, x_n]$

Consider

$$0 \to Q \to R^r \xrightarrow{f} I \to 0$$
$$0 \to Q' \to R'^r \xrightarrow{f'} I' \to 0$$

and tensor (*) with R'^r, R^r and B' over C' to obtain the following diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Q' \arrow[r] \arrow[d] & Q \arrow[d] \arrow[dddll, bend left, "f_0"] & \\
0 \arrow[r] & R_0^r \otimes_k J \arrow[r] \arrow[d] & R'^r \arrow[r] \arrow[d, "f'"] & R^r \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & R_0 \otimes_k J \arrow[r] \arrow[d, "g"] & R' \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
 & B_0 \otimes_k J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

(Idea: we are trying to construct B' s.t. $B_0 \otimes_k J \hookrightarrow B'$. So we want to lift generators of I to R'.)

Snake lemma. $\exists\ \delta_0 : Q \to B_0 \otimes_k J$ s.t.
$B_0 \otimes_k J \hookrightarrow B' \iff \delta_0 = 0$.

Notice that the ideal $F_0 = \langle f(a)b - f(b)a \mid a, b \in R^r \rangle$, $F_0 \subseteq Q$, and these relations can always be lifted. More specifically, if $I = \langle f_1, \dots, f_r \rangle$ and we pick liftings $f_1', \dots, f_r'$, the relation $f_i e_j - f_j e_i$ in Q always lifts to $f_i' e_j' - f_j' e_i' \in Q'$. Thus, δ_0 factorizes through Q/F_0.

and then we obtain $\delta_1 \in \mathrm{Hom}_{C'}(Q/F_0, B_0 \otimes_k J)$.

Recall.

$$T^2(B/C, B_0 \otimes J) \simeq \mathrm{coker}\left(\mathrm{Hom}(R^r/IR^r, B_0 \otimes J) \to \mathrm{Hom}(Q/F_0, B_0 \otimes J)\right).$$

Thus, the previous choice of generators defines an element

$$\delta \in T^2(B/C, B \otimes J) \simeq T^2(B_0/k, B_0 \otimes J)$$

Claim. δ is indep. of the choices of liftings.

Take $f_i'' \in R'$ another set of liftings of f_i.

Since $f_i' - f_i''$ goes to zero under $R' \to R$,

we obtain a map $R'^r \to R_0 \otimes J$, and the composition

$$R^r \hookrightarrow R'^r \to R_0 \otimes J \to B_0 \otimes J$$

factorizes through F/IF. Thus the difference $\delta_0' - \delta_0''$ goes to zero in T^2.

Also, we need to check δ is independent of $R \twoheadrightarrow B$ (to be checked).

Now, we want to see δ means an obstruction for the existence of B', i.e., B'/C' $\exists \iff \delta_0 = 0$.

If B'/C' exists $\overset{}{\underset{\text{flatness}}{\Longrightarrow}} \delta_0 = 0 \implies \delta = 0$.

Assume $\delta = 0$. By def., $\delta_1 \in \mathrm{Hom}(Q/F_0, B_0 \otimes J)$ lifts to $\gamma : R^r/IR^r \to B_0 \otimes J$ i.e., if $\pi : Q/F_0 \to R^r/IR^r$, then $\gamma \circ \pi = \delta_1$. Now, since R^r is free,

$$\begin{tikzcd}
R_0 \otimes_k J \arrow[r, two heads, "g"] & B_0 \otimes_k J \arrow[r] & 0 \\
\exists \tilde{\gamma} & R^r \arrow[ul, dashed] \arrow[u, "\gamma"'] &
\end{tikzcd}$$

and $\tilde{\gamma}$ is determined by choice $g_1, \dots, g_r \in R_0 \otimes_k J$.

Take $\tilde{f}_i = f'_i - g_i$ and define $\tilde{I} = \langle \tilde{f}_1, \dots, \tilde{f}_r \rangle$

original generators

$\rightsquigarrow \tilde{B} = R'/\tilde{I}$. We compute that $\tilde{\delta}_0 = 0$

$\Rightarrow \tilde{B}$ flat over C'. Notice that the original f' was $e_i \mapsto f'_i$, and now we change f' by $\tilde{f} : e_i \mapsto \tilde{f}_i := f'_i - g_i$. Then, for $q = \sum a_i e_i \in Q$,

$$\tilde{f}(q) = \sum a_i (\tilde{f}_i - g_i) = f'(q) - \underbrace{\sum a_i g_i}_{c}$$

$\rightsquigarrow \tilde{\delta}_0(q) = \delta_0(q) - c$, with c the image of $\sum a_i g_i$ in $B_0 \otimes J$, but

$$\tilde{\delta}_0(q) = \delta_0(q) - \tilde{\gamma}(g(q)) = \delta_0(q) - \gamma(q) = \delta_0(q) - \delta_0(q)$$
$$= 0$$

Talk 4 : Defs. are a torsor under $\mathrm{Hom}(I/I^2, B\otimes J)$.

Talk 3: (1) Different choices are parametrized by $\mathrm{Hom}(\Omega_{R/C}, B\otimes J)$.

Thus, equiv. classes is a torsor under

$$\mathrm{coker}\big(\mathrm{Hom}(\Omega_{R/C}, B\otimes J)\to \mathrm{Hom}(I/I^2, B\otimes J)\big) =: T^1(B/C, B\otimes J) \simeq T^1(B_0/k, B_0\otimes J),$$

(2) Group of automorphisms of B' lifting id_B is isomorphic to $T^0(B_0/k, B_0\otimes J)$.

Remark. (1) X_0 is a l.c.i $\iff Q = F_0$.

Then, $\mathrm{Hom}(Q/F_0, B_0) = 0 \Rightarrow T^2 = 0$ and there are no obstructions.

(2) Over X_0/k non-affine, local modules T^i can be glued on sheaves $\mathcal{T}^i_{X_0} := T^i_{X_0}(X_0/\mathrm{Spec}\, k, \mathcal{O}_{X_0})$.

If X_0 is normal and reduced,

$$\mathcal{T}^1_{X_0} \simeq \mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}), \quad \mathcal{T}^2_{X_0} \simeq \mathcal{E}xt^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0})$$

(see Schlessinger, "On rigid singularities").

In the global setting we have

Thm. (a) $\exists$ 3 succesive obstructions for the existence X' extension of X over C', lying on

$H^0(X_0, \mathcal{T}^2_{X_0}\otimes J)$, $H^1(X_0, \mathcal{T}^1_{X_0}\otimes J)$ and $H^2(X_0, \mathcal{T}^0_{X_0}\otimes J)$.

(b) Let $\mathrm{Def}(X/C, C') = \{\text{extensions } X'/C'\}/\simeq$. Then

$$0 \to H^1(X_0, T^0_{X_0} \otimes J) \to \mathrm{Def}(X/C, C') \to H^0(X_0, T^1_{X_0} \otimes J)$$
$$\to H^2(X_0, T^2_{X_0} \otimes J) \quad \text{is exact.}$$

(c) Given X'/C' ext. of X/C, $H^0(X_0, T^0_{X_0} \otimes J)$ is the group of automorphisms of X'/C' lifting the identity of X/C.

Proof. Write $X = \bigcup \mathcal{U}_i$, $\mathcal{U}_i$ affine, so $\exists$ an obstruction in $H^0(\mathcal{U}_i, T^2_{\mathcal{U}_i} \otimes J)$ for the existence of $\mathcal{U}_i'$, so there is a global obstruction

$\delta_1 \in H^0(X_0, T^2_{X_0} \otimes J)$ global obstruction. If $\delta_1 = 0$, then $\exists$ $\mathcal{U}_i'$ local extensions.

On the overlaps $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ we have two defs.
$\mathcal{U}_i'|_{\mathcal{U}_{ij}}, \mathcal{U}_j'|_{\mathcal{U}_{ij}} \longleftrightarrow$ 2 elements in $H^0(\mathcal{U}_{ij}, T^1_{\mathcal{U}_{ij}} \otimes J)$
and their diff d_{ij} is in $H^0(\mathcal{U}_{ij}, T^1_{\mathcal{U}_{ij}} \otimes J)$. We have
$\mathcal{U}_{ij} \rightsquigarrow \delta_{ij}$, $\mathcal{U}_{ik} \rightsquigarrow d_{ik}$, $\mathcal{U}_{jk} \rightsquigarrow \delta_{jk}$ and by def.

$\delta_{ij} - \delta_{ik} + \delta_{jk} = 0 \rightsquigarrow$ they define

$$\delta_2 \in H^1(X_0, T^1_{X_0} \otimes J).$$

- If $\delta_2 = 0$, the extension $\mathcal{U}_i'$ are equiv. on $\mathcal{U}_{ij}$, so we can choose $\varphi_{ij} : \mathcal{U}_i'|_{\mathcal{U}_{ij}} \xrightarrow{\sim} \mathcal{U}_j'|_{\mathcal{U}_{ij}}$ and they induces sections in $H^0(\mathcal{U}_{ijk}, T^0_{\mathcal{U}_{ijk}} \otimes J)$ that agree in 4-intersections $\rightsquigarrow$ $\delta_3 \in H^2(X_0, T^0_{X_0} \otimes J)$.

(b) Take X'/C' and $X = \bigcup \mathcal{U}_i$, $\mathcal{U}_i$ affine

$\rightsquigarrow$ element in $H^0(\mathcal{U}_i, \mathcal{T}^1_{\mathcal{U}_i} \otimes J) \rightsquigarrow$ element in $H^0(X_0, \mathcal{T}^1_{X_0} \otimes J)$.

Now, given an element an element in $H^0(X_0, \mathcal{T}^1_{X_0} \otimes J)$ we have local defs., and the obstruction to glue them lives in $H^2(X_0, \mathcal{T}^0_{X_0} \otimes J)$

$$\Rightarrow \mathrm{Def}(X'/C', C) \to H^0(X_0, \mathcal{T}^1_{X_0} \otimes J) \to H^2(X_0, \mathcal{T}^0_{X_0} \otimes J).$$

If X_1', X_2' give the same $\delta \in H^0(X_0, \mathcal{T}^1_{X_0} \otimes J)$

$\Rightarrow \forall\, \mathcal{U}_i$, $\varphi : X_1'|_{\mathcal{U}_i} \xrightarrow{\sim} X_2'|_{\mathcal{U}_i} \rightsquigarrow \varphi_{ij} = \varphi_j^{-1} \circ \varphi_i$ aut. of $X_1'|_{\mathcal{U}_{ij}} \rightsquigarrow$ element in $H^0(\mathcal{U}_{ij}, \mathcal{T}^0_{\mathcal{U}_{ij}} \otimes J)$ s.t. they agree in 3-intersections. ∎

Corollary. If X_0 is non-singular, then

(a) $\exists$ only one obstruction in $H^2(X_0, \mathcal{T}_{X_0} \otimes J)$ for existence of X'/C'.

(b) If extension exists, equiv. classes are a torsor under $H^1(X_0, \mathcal{T}_{X_0} \otimes J)$.

Proof. X_0 non-singular $\Rightarrow \mathcal{T}^1_{X_0} = \mathcal{T}^2_{X_0} = 0$ and

$\mathcal{T}^0_{X_0} \simeq \mathcal{T}_{X_0} \leftarrow$ tangent sheaf.

∎