RSVW formula

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$\boxed{\text{Geometric } \text{~~Intuitive~~ interpretation of scattering amplitudes}}$

(via algebraic curves)

Basics of twistor theory (revision)

infinity twistor

$$\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$$

$$\boxed{\text{spacetime}}^{\,\mathbb{M}_{\mathbb{C}}} \longleftrightarrow \boxed{\text{twistor space}}$$

(coords $x^{\alpha\dot\alpha}$) — incidence relations — (coords $\underbrace{(\mu^{\dot\alpha}, \lambda_\alpha)}_{Z^A}$)

! spinor-helicity formalism

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$$

point $\longleftrightarrow$ Riemann sphere $\mathbb{CP}^1$

α-plane $\longleftrightarrow$ point
(2D null plane)

solutions to ZRM eq. (e.g. solutions to $\partial^{\alpha_1\dot\alpha}\phi_{\alpha_1\alpha_2\ldots\alpha_{2h}} = 0$) $\longleftrightarrow$ (Penrose transform) cohomology classes $(H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2)))$

- supertwistor space:
  - ↳ we just add anti-commuting Grassmann variables $(\mu^{\dot\alpha}, \lambda_\alpha, \chi^a)$
  - ↳ add incidence relations for them:

$$\chi^a = \eta^{a\alpha}\lambda_\alpha$$

  - ↳ our cohomology class representative from Penrose transform will be:

$$a(z) \longmapsto A(z,\chi) = a(z) + \chi^a \ldots + \ldots + \chi^4 b(z)$$

we can see
these as representatives
for (+)-helicity gluons

negative
hel. gl.

↳ why we therefore include $\mathcal{N}=4$ SYM: effectively we pack all the amplitudes (connected to different helicity combinations) into one object

super-amplitude

then we can extract individual ones by looking at powers of χ^a

RSVW formula

↳ formula for tree-level amplitude (in trivial bcg) for any combination of positive /negative helicity gluons

↳ version for $\mathcal{N}=4$ SYM: (for $d+1$ negative helicity gluons)

$$\mathcal{A}^{\mathcal{N}=4}_{n,d} = \int \frac{d^{4|4(d+1)}\mathcal{U}}{\text{vol } GL(2,\mathbb{C})} \prod_{i=1}^{n} \frac{D\sigma_i}{(i\ i{+}1)} A_i(\mathcal{Z}(\sigma_i))$$

↳ ingredients:

how many preimages any point has

· map $\mathcal{Z}: \mathbb{CP}^1 \to \mathbb{PT}$ (degree d holomorphic map)

$$\mathcal{Z}^A(\sigma) := \mathcal{U}^A_{\underline{a_1} \dots \underline{a_d}} \underbrace{\sigma^{\underline{a_1}} \dots \sigma^{\underline{a_d}}}_{d}$$

$\in SL(2,\mathbb{C})$

symmetry: $\sigma^{\underline{a}} \to \Omega^{\underline{a}}{}_{\underline{b}}\, \sigma^{\underline{b}}$

rescaling sym.: $\mathcal{Z}^A(\sigma) \sim r\, \mathcal{Z}^A(\sigma)$
(because we are

by definition

in twistor space) of twistor space

- integral over twistor space & measure:

$$\mathcal{U} = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \\ \chi^a \end{pmatrix}$$ (for each coeficient ...)

- cohomology classes $A_i(Z(\sigma_i))$ for each gluon
- $D\sigma_i = \varepsilon^{\underline{ab}} \sigma_{i\underline{a}} d\sigma_{i\underline{b}} = (\sigma_i d\sigma_i)$
- vol $GL(2,\mathbb{C})$ ...

↳ version for non-super-symmetry: (we chose coef. in front of power of Grasmann var.)

$$\mathcal{A}_{n,d} = \int \frac{d^{4(d+1)} \mathcal{U}}{\mathrm{vol}(GL(2,\mathbb{C}))} |g^-|^4 \prod_{i=1}^{n} \frac{D\sigma_i}{(i\ i+1)} \prod_{j\in g^+} a_j(Z(\sigma_j)) \prod_{k\in g^-} b_k(Z(\sigma_k))$$

some determinant connected to negative helicity gluons

positive helicity gluons

negative hel. gluons

Key point of the formula

- we can express amplitudes in 4d YM theory just using maps on $\mathbb{CP}^1$
- seems like we have the following equivalence:

tree-level YM amplitudes (for $\mathcal{N}=4$) in 4D Minkowski space $\longleftrightarrow$ correlation functions in free 2D theory on $\mathbb{CP}^1$

- geometric interpretation:

↳ Z gives us some polynomial on twistor space = connected holomorphic curve of degree d

see it in the same sense as $\mathbb{CP}$'s are twistor lines

↳ amplitudes have support (are nonzero) only on these curves

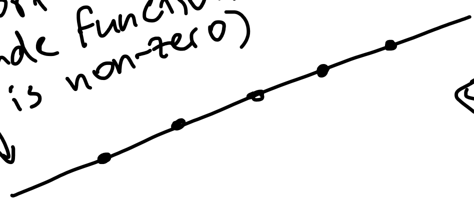
↳ <u>Example</u>: 5-point amplitude

degree of z

(1) helicities $= (+++--) \Rightarrow d = 1$

curve of degree 1 = line

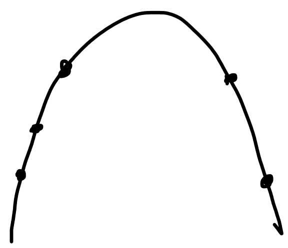
curve = support of amplitude function (where it is non-zero)



← all points (gluons) on one line (Riemann sphere)

(2) helicities $= (++---) \Rightarrow d = 2$

connected curve of degree 2 = "parabola"



menda tudi Riemannova sfera

(za $d=3$ je pa kompleksni torus)

- Pazi! Vse te črte so zares projektivne kompleksne stvari.
  Npr. premica je Riemannova sfera etc

- Poleg tega to vložimo v twistor prostor

<u>Fun fact</u>: How do we see that $\mathcal{A}(+++\ldots) = 0$ or $\mathcal{A}(-++\ldots) = 0$?

↳ let's look at degree of corresponding curves?

- one $\ominus$: $d = 1 - 1 = 0$

  curve = point

  all points on curve = all points are the same

  $\lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}}$

$\Rightarrow \; p_1 = p_2 = \ldots \;\Rightarrow\; \lambda_1 \propto \lambda_2 \propto \ldots$

$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \ldots$

(case $n=3$ is special because we can get $p_1 = p_2 = p_3$ with requiring only one of $\begin{cases} \lambda_1 \propto \lambda_2 \propto \lambda_3 \\ \tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3 \end{cases}$ to hold

complex plane is defined by three-points so condition $\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$ comes for free)

(že če imaš pa samo eno točko več, pa ne moreš vseh vezi dati v $\tilde{\lambda}$ spr.; ostane ti sicer le ena na kateri imaš vezi, ampak $p_1 = p_2 = p_3 = \ldots$ pomeni, da imaš zares $n-1$ vezi povezanih s to točko: $p_i = p_1$; $p_i = p_2$ itd)

- all $\oplus$: $d = 0 - 1 = -1$

  no such known curves...

String theory interpretation

- connected holomorphic curve embedded in $\mathcal{N}=4$ SYM twistor space is in fact so called <u>D-instanton</u>
- D instanton = brane with Dirichlet boundary conditions along all non-compact dimensions, including Euclidean time

  (we consider corresponding object in twistor space)

  $\updownarrow$

  equivalent instantons in QFT (e.g. in Dunajski's book)

Dunajski's book)

- we normally get these instantons by doing integrals over moduli spaces of Riemann surfaces

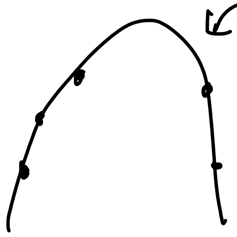
integrals over twistor space?

we indeed have moduli space because of $SL(2,\mathbb{C})$ symmetry

in our case $\mathbb{CP}^{3|4}$

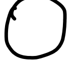
- which thing exactly is instanton?

this curve



Summary & why was crucial we used twistor space (and not Minkowski)

- we want $SL(2,\mathbb{C})$ symmetry (symmetry of Riemann sphere)
- we therefore wish to have theory on this sphere
- but in Minkowski space: each particle is inserted in one point
- we therefore go to twistor space where any point is described with Riemann sphere worth of points
- twistor space provide us with ambient space of these Riemann sphere, so we don't need to see maps on $\mathbb{CP}^1$ as maps, but rather as deformed $\mathbb{CP}^1$s

(e.g.  →  for third order)

- amplitude setting?

  Inserted points on our manifold
  (= deformed $\mathbb{CP}^1$)

- how do we get amplitude (= scalar)?
  ↳ sum / integrate over all possible such configuration in twistor space

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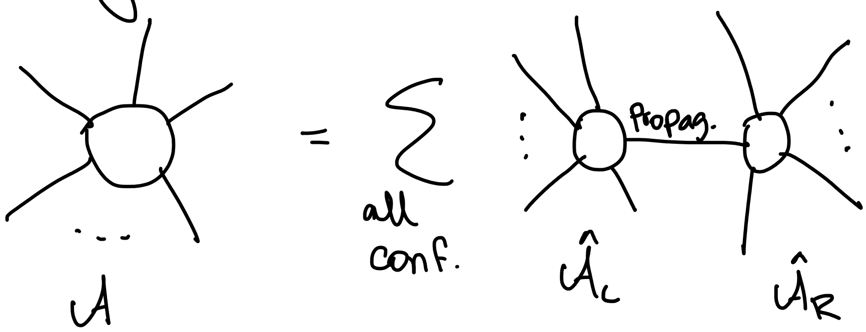
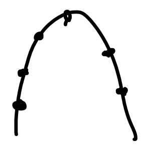
Shematic proof of RSVW formula

- as crucial property of amplitudes we will use BCFW recursion:

$$\mathcal{A} = \sum_{\substack{\text{all}\\ \text{configurations}}} \frac{\hat{\mathcal{A}}_L \, \hat{\mathcal{A}}_R}{p^2}$$
(variables shifted)

shematically:



- in terms of curves from before:
- Example 6-point (+++−−−) amplitude ↗ 

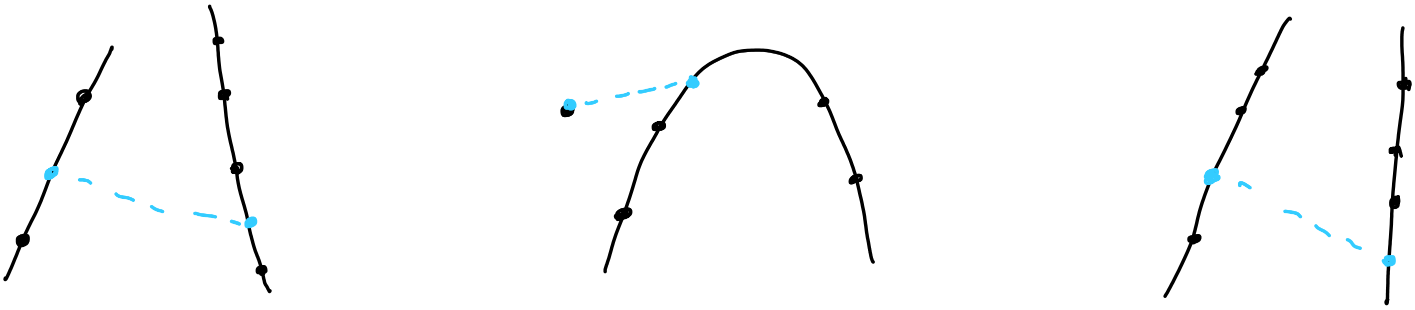
parabola (d=2)

BCFW recursion produces: $\boxed{d = \#\,\text{negative hel. gl.} - 1}$

(1) 3-point (+, −−)
5-point (−, +++−)

(2) 3-point (+, +−)
5-point (−, ++−−)

(3) 4-point (+, +−−)
4-point (−, ++−)



we can also notice:

$d_1=1, d_2=1 \Big\} d_1+d_2=2=d$ $\qquad$ $d_1=0, d_2=2 \Big\} d_1+d_2=d$ $\qquad$ $d_1=1, d_2=1 \Big\} d$

- we could in fact tell the possible splittings only from all possible curves!

Can we use BCFW recursion to prove RSVW formula?

- recursion: (1) check for 3-point seed amplitudes ($(+--)=\text{MHV}$; $(++-)=\overline{\text{MHV}}$) if formula reduces to correct expressions (Parke-Taylor)

  (2) prove recursion expression

  (3) (check it is well-defined)

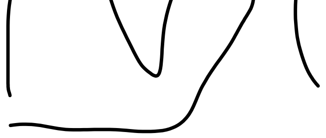
- let's focus on (2):

  ↳ main problem: how do we split one curve into two curves with lower degree ($d=d_1+d_2$)

Intermezzo:

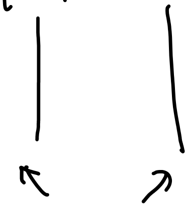
consider space of (connected) curves:

what happens if we consider a sequence of curves like this:



(moduli space)

in "limit":



two separate curves!

note also that the degree of the two curves is less than degree of initial curve (e.g. parabola will look like two lines)

this is exactly what we had above!

(in mathematics: closure of moduli space = boundary divisor)

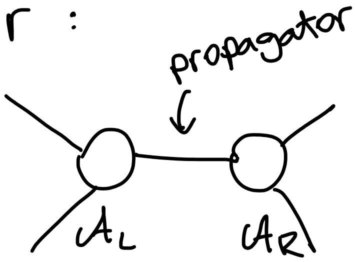
- which curves are therefore secretly two curves?
  - ↳ the ones with singularities! (the curve is disconnected in singular point)

Let's return to our case!

↳ if we want our curves to secretly consist of two curves, they better have poles

↳ for YM, poles are exactly propagators (see expression for BCFW recursion)

↳ remember:



since we cut the curve exactly at the propagator, the newly formed two curves are indeed connected to A_L and A_R (subamplitudes)

↳ we therefore proved recursion relation

□

What did we add to our understanding? insights

↳ our maps producing deformed Riemann spheres are polynomials (degree of polynomial = # neg. hel. gl. − 1)

↳ are constructed from polynomial curves of lower order!