Chapter 9

Thursday, 19 February 2026 16:57

Consider $E = \int d^d x \left[|F|^2 + |D\phi|^2 + U(\phi) \right] = E_F + E_\phi + E_U$

Let $\phi_c = \phi(cx)$, $A_c = cA_c(x)$, $F_c(x) = c^2 F(cx)$, $D_c \phi_c = cD\phi(cx)$

So $E_c = c^{-(d-4)} E_F + c^{-(d-2)} E_{D\phi} + c^{-d} E_U$

$E'(c) = 0 \Rightarrow \quad (d-4)E_F + (d-2)E_{D\phi} + dE_U = 0$

If $d = 2$ $\quad E_F = E_U$ is a Solution

The Abelian-Higgs model

Let $\phi : \mathbb{R}^{2+1} \to \mathbb{C}$, $A : \mathbb{R}^{2+1} \to \Lambda^1$ is a $U(1)$ Valued 1 form

and $D\phi \equiv d\phi - iA\phi$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi \bar{D}^\mu \bar{\phi} - \frac{c}{8}\left(1 - |\phi|^2\right)^2$$

L is Invariant Under the local $U(1)$ transformation

$$\phi \mapsto e^{i\alpha}\phi, \; A \mapsto A + d\alpha. \; D\phi \to e^{i\alpha} D\phi$$

Where $\alpha : \mathbb{R}^{2,1} \to \mathbb{R}$

Our E.L eqs are

$$\partial_\mu F^{\mu\nu} = \frac{i}{2}\left(D^\nu \phi \, \bar{\phi} - \phi \bar{D}^\nu \bar{\phi} \right)$$

$$D_\mu D^\mu \phi = \frac{c}{2}\left(1-|\phi|^2\right)\phi$$

A Static Vacuum Solution is Unique upto a gauge Where we have

$$|\phi|=1,\ D\phi=0,\ F_{12}=\partial_1 A_2-\partial_2 A_1=0$$

If we gauge fix such that $\phi\in\mathbb{R}^+$, $\phi=1$ leads to SSB Where our gauge boson acquires a mass.

To See this, consider the Lagrangian around $\phi=1$ So we have

$$L=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}A_\mu A^\mu$$

Where we Can identify $M_A^2=1$

Linearising the first E.L eq around $\phi=1$ we find

$$\phi=1+\eta$$

$$\partial_\mu F^{\mu\nu}=\frac{i}{2}\left(\left(\partial^\nu-iA^\nu\right)(1+\eta)\left(1+\bar{\eta}\right)-(1+\eta)\left(\partial^\nu+iA^\nu\right)\left(1+\bar{\eta}\right)\right)$$

$$=\frac{i}{2}\left(\left(-iA^\nu+\bar{D}^\nu\bar{\eta}\right)\left(1+\bar{\eta}\right)-(1+\eta)\left(iA^\nu+D^\nu\bar{\eta}\right)\right)$$

$$=\frac{i}{2}\left(-iA^\nu-iA^\nu\eta+\bar{D}^\nu\bar{\eta}-iA^\nu-\bar{\eta}iA^\nu-D^\nu\eta\right)+O(\eta^2)$$

$$\simeq\frac{i}{2}\left(-2iA^\nu+\bar{D}^\nu\bar{\eta}-D^\nu\eta-iA^\nu(\eta+\bar{\eta})\right)$$

So at $\eta=0$

$$\partial_\mu F^{\mu\nu}=A^\nu$$

$$\partial_\mu\partial^\mu A^\nu-\partial_\mu\partial^\nu A^\mu=A^\nu$$

$$\Rightarrow \partial^2 A^\nu-\partial^\nu\partial\cdot A=A^\nu$$ Choose $\partial\cdot A=0$ gauge

$$\partial^2 A^\nu - A^\nu = 0$$

$$(\partial^2 - 1)A^\nu = 0 \quad \Rightarrow \quad M_A^2 = 1$$

We are interested in t independent solutions in $d=2$ so

$$\tfrac{1}{2}F_{ij}F^{ij} = B^2, \quad D_i\phi \bar{D}_i\bar{\phi} = D_1\phi\bar{D}_1\bar{\phi} + D_2\phi\bar{D}_2\bar{\phi},$$

with $B = F_{12} = -F_{12}$

We have the Energy functional

$$E[A,\phi] = \int_{\mathbb{R}^2}\left(\frac{1}{2}B^2 + \frac{1}{2}|D\phi|^2 + \frac{C}{8}(1-|\phi|^2)^2\right)d^2x$$

Solitons in this model are called vortices.

$C > 1$ describes type II superconductors where vortices repel

$C < 1$ describe type I superconductors where vortices attract

$C = 1$ is called critical coupling where no attraction/repulsion occurs.

Vortex number

Move to polar coords

Let $A = A_r dr + A_\theta d\theta$, $A_r = \frac{1}{r}(A_1 x^1 + A_2 x^2)$, $A_\theta = A_2 x^1 - A_1 x^2$

$$r^2 = x^2 + y^2 \Rightarrow 2r\,dr = 2x\,dx + 2y\,dy \Rightarrow dr = \frac{1}{r}(x\,dx + y\,dy)$$

$$\theta = \arctan\left(\frac{y}{x}\right) \Rightarrow d\theta = \frac{-y x^{-2}}{1+(\frac{y}{x})^2}dx + \frac{x^{-1}}{1+(\frac{y}{x})^2}dy$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \frac{1}{r^2}\left(-y\,dx + x\,dy\right)$$

Using $x = r\cos\theta$, $y = r\sin\theta$, $dx = \cos\theta\,dr - r\sin\theta\,d\theta$, $dy = \sin\theta\,dr + r\cos\theta\,d\theta$

$$A = A_x(\cos\theta\,dr - r\sin\theta\,d\theta) + A_y(\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= (A_x\cos\theta + A_y\sin\theta)dr + r(A_y\cos\theta - A_x\sin\theta)d\theta$$

$$= A_r\,dr + A_\theta\,d\theta$$

$$A_x\cos\theta + A_y\sin\theta = \frac{1}{r}(A_x x + A_y y)$$

$A_y r\cos\theta - A_x r\sin\theta = A_y x - A_1 y$ as expected.

Our curvature tensor is

$$F = F_{r\theta}\,dr\wedge d\theta \quad \text{with} \quad F_{r\theta} = r F_{12}$$

$F = F_{12}\,dx^1\wedge dx^2$, $dx^1\wedge dx^2 = r\,dr\wedge d\theta$

$F = F_{12}\,r\,dr\wedge d\theta \Rightarrow rF_{12} = F_{r\theta}$

$F = F_{r\theta}\,dr\wedge d\theta$ as desired.

We want finite energy solutions so we impose

$$B \to 0, \quad |\phi| \to 1, \quad D\phi \to 0 \quad \text{as} \quad r \to \infty.$$

Choose the radial gauge $A_r = 0$ so that the limit

$$\phi_\infty(\theta) = \lim_{r\to\infty} \phi(r,\theta)$$

exists and $|\phi_\infty| = 1$. In this gauge $D_r\phi = \partial_r\phi - iA_r\phi = \partial_r\phi \to 0$ as $r \to \infty$.

By choosing $A_r = 0$, $D_r \phi = \partial_r \phi \to 0$ as $r \to \infty$, this lets the limit exist.

Under $A \mapsto A + d\alpha$, $A_r \mapsto A_r + \partial_r \alpha$, $A_\theta \mapsto A_\theta + \partial_\theta \alpha$

$A_r = 0 \Rightarrow \partial_r \alpha = 0$ so $\alpha = \alpha(\theta)$.

If we want this to be defined on the whole plane including $(0,0)$

requires $\alpha =$ constant.

Since $\phi_\infty(\theta) = \lim_{r\to\infty} \phi(r,\theta)$, $|\phi_\infty(\theta)| = 1$

$\phi_\infty(\theta) \in S^1 \Rightarrow \phi_\infty(\theta) = e^{i\chi_\infty(\theta)}$

This means $\chi_\infty : S^1_\infty \to S^1 \subset \mathbb{C}$

i.e χ_∞ is mapping the circle at inf to the unit circle in the internal space

of the Higgs field.

The topological charge N is defined as the winding number of χ_∞

$$\theta \mapsto \theta + 2\pi, \quad \chi_\infty(\theta) \mapsto \chi_\infty(\theta) + 2\pi N$$

So ϕ_∞ is single valued. N does not change under cont. finite energy

deformations or time evolution.

We can also compute the topological charge with

$$\text{Flux} = \int_{\mathbb{R}^2} F = \lim_{r\to\infty} \int_0^{2\pi} A_\theta \, d\theta = \int_0^{2\pi} \frac{\partial \chi_\infty}{\partial \theta} d\theta = \chi_\infty(2\pi) - \chi_\infty(0) = N$$

or alternatively

$$N = \int_0^{2\pi} \chi_\infty^* d\theta = \int_0^{2\pi} d\chi_\infty = \int_0^{2\pi} \frac{\partial \chi_\infty}{\partial \theta} d\theta = \chi_\infty(2\pi) - \chi_\infty(0)$$

Example for $N=1$

Choose $A_r=0$, $A=f(r)d\theta$, $\phi=h(r)e^{i\theta}$

The field eqs give

$$h''+\frac{1}{r}h'-\frac{1}{r^2}(1-f)^2h+\frac{c}{2}(1-h^2)h=0$$

$$f''-\frac{1}{r}f'+(1-f)h^2=0$$

Bogomolny equations

Choose $c=1$

Theorem 9.2.1

The GL energy functional with $c=1$ is bounded from below by

$$E\geq\pi N$$

with saturation iff

$$D_1\phi+iD_2\phi=0,\quad B-\frac{1}{2}(1-|\phi|^2)=0$$

Proof:

$$E[A,\phi]=\frac{1}{2}\int_{\mathbb{R}^2}\left[\left(B-\frac{1}{2}(1-|\phi|^2)\right)^2+\left(\overline{D_1\phi}-i\overline{D_2\phi}\right)\left(D_1\phi+iD_2\phi\right)\right]d^2x$$

$$+\frac{1}{2}\int_{\mathbb{R}^2}\left[B(1-|\phi|^2)+i\overline{D_2\phi}D_1\phi-i\overline{D_1\phi}D_2\phi\right]d^2x$$

Using $\partial_2(\bar\phi D_1\phi)=\overline{D_2\phi}\,D_1\phi+\bar\phi D_2D_1\phi$

$\partial_2\bar\phi=\overline{D_2\phi}-iA_2\bar\phi$

$\partial_2(D_1\phi)=\partial_2(\partial_1-iA_1)\phi=(\partial_2\partial_1-i\partial_2A_1-iA_1\partial_2)\phi$

$$D_\mu D_\nu \phi = D_\mu(\partial_\nu \phi - iA_\nu \phi)$$

$$= (\partial_\mu - iA_\mu)(\partial_\nu \phi - iA_\nu \phi)$$

$$= \partial_\mu \partial_\nu \phi - i\partial_\mu A_\nu \phi - iA_\nu \partial_\mu \phi - iA_\mu \partial_\nu \phi - A_\mu A_\nu \phi$$

$$= \partial_\mu(\partial_\nu \phi - iA_\nu \phi) - iA_\mu(\partial_\nu \phi - iA_\nu \phi)$$

$$= \partial_\mu D_\nu \phi - iA_\mu D_\nu \phi$$

So $\partial_2(\bar{\phi} D_1 \phi) = \partial_2 \bar{\phi} D_1 \phi + \bar{\phi} \partial_2 D_1 \phi$

$$= (\overline{D_2 \phi} - \cancel{iA_2 \bar{\phi}}) D_1 \phi + \bar{\phi}(D_2 D_1 \phi + \cancel{iA_2 D_1 \phi})$$

$$= \overline{D_2 \phi}\, D_1 \phi + \bar{\phi} D_2 D_1 \phi \qquad \text{as desired}$$

and using $[D_1, D_2]\phi = -iB\phi$

$$[\partial_1 - iA_1, \partial_2 - iA_2]\phi = \cancel{[\partial_1, \partial_2]}\phi + [\partial_1, -iA_2]\phi + [-iA_1, \partial_2]\phi + \cancel{[-iA_1, -iA_2]}\phi$$

$$= -i\partial_1(A_2\phi) + iA_2\partial_1\phi - iA_1\partial_2\phi + i\partial_2(A_1\phi)$$

$$= -i\partial_1 A_2 \phi + i\partial_2 A_1 \phi$$

$$= -i(\partial_1 A_2 - \partial_2 A_1)\phi$$

$$= -iB\phi$$

So our second integral

$$\frac{1}{2}\int_{\mathbb{R}^2} d^2x \left[B - i\bar{\phi} D_1 D_2 \phi + i\bar{\phi} D_2 D_1 \phi + i\overline{D_2 \phi} D_1 \phi - i\overline{D_1 \phi} D_2 \phi \right]$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} \left[B + i\partial_2(\bar{\phi} D_1 \phi) - i\partial_1(\bar{\phi} D_2 \phi) \right] d^2x$$

The second and third term vanish by Greens theorem and

$\int \frac{1}{2} B^2 d^2x$ is just the flux $\cdot \frac{1}{2}$ so πN we finally find

$\int_{\mathbb{R}^2} \frac{1}{2} B^2 \, d^2x$ is just the flux $\cdot \frac{1}{2}$ so πN. We finally find

$$E = \frac{1}{2}\int_{\mathbb{R}^2} \left[\left(B - \frac{1}{2}(1-|\phi|^2)\right)^2 + |D_1\phi + iD_2\phi|^2 \right] d^2x + \pi N.$$

Taubes equation

Let $\phi = e^{u/2 + i\chi}$ where u, χ are real valued and u is gauge inv.

Use the first Bogomolny equation $D_1\phi + iD_2\phi = 0$

$$\partial_1 \phi = \left(\frac{1}{2}\frac{\partial u}{\partial x} + i\frac{\partial \chi}{\partial x}\right)e^{u/2+i\chi}, \quad iA_1\phi = iA_1 e^{u/2+i\chi}$$

$$i\partial_2 \phi = \left(\frac{i}{2}\frac{\partial u}{\partial y} - \frac{\partial \chi}{\partial y}\right)e^{u/2+i\chi}, \quad -A_2\phi = -A_2 e^{u/2+i\chi}$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + i\frac{\partial \chi}{\partial x} + iA_1 + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{\partial \chi}{\partial y} - A_2 = 0$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} - \frac{\partial \chi}{\partial y} - A_2 + i\left(\frac{1}{2}\frac{\partial u}{\partial y} + \frac{\partial \chi}{\partial x} + A_1\right) = 0$$

$$\Rightarrow A_1 = -\frac{1}{2}\frac{\partial u}{\partial y} + \frac{\partial \chi}{\partial x}$$

$$A_2 = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{\partial \chi}{\partial y}$$

So $A = A_1 dx + A_2 dy$

$$\Rightarrow A = \frac{1}{2}\left(\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx\right) + d\chi$$

$B = \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}\Delta_0 u$ where $\Delta_0 = \partial_1^2 + \partial_2^2$

The Second Bogomolny eq imposes

$$\Delta_0 u - e^u + 1 = 0$$

which is valid outside of the zeroes of ϕ where u has log singularities.

If $\phi(\underline{X}) = 0$ is a zero of order K then $\phi \sim \text{const} \cdot |\underline{x} - \underline{X}|^K$

near this zero so this eq needs to be supplemented by B.C

$$u \sim 2K \ln|\underline{x} - \underline{X}| + \dots$$

$$u \to 0 \quad \text{as} \quad |\underline{x}| \to \infty$$

$|\phi| = e^{u/2} \Rightarrow u = 2\ln|\phi|$, If ϕ has order K zero $\phi(\underline{x}) = (\underline{x} - \underline{X})^K h(\underline{x})$

$|\phi(\underline{x})| = |\underline{x} - \underline{X}|^K |h(\underline{x})| \Rightarrow \ln|\phi| = K\ln|\underline{x} - \underline{X}| + \ln|h(\underline{x})|$

$\Rightarrow u = 2K\ln|\underline{x} - \underline{X}| + 2\ln|h(\underline{x})|$

near zero $u \sim 2K\ln|\underline{x} - \underline{X}|$

Can encode the first B.C by considering $\Delta_0 u \sim \Delta_0 2K\ln|\underline{x} - \underline{X}|$

$\Delta \ln|\underline{x}| = 2\pi\delta(\underline{x})$

$$2K\Delta_0 \ln|\underline{x} - \underline{X}| = 4\pi K\delta^2(\underline{x} - \underline{X})$$

To include these zeroes, the equation has to be modified as

$$\Delta_0 u - e^u + 1 = 4\pi \sum_s K_s \delta(\underline{x} - \underline{X}_s)$$

where K_s are the multiplicities of the zeroes.

Example:

N = 1 Vortex

$\Delta_0 u - e^u + 1 = 0$, Let $u = u(r)$

Then $\Delta_0 u = u''(r) + \frac{1}{r} u'(r)$ so

$u''(r) + \frac{1}{r} u'(r) + 1 - e^u = 0$

We have BC's

$u = 2\ln r + u_0 + u_1 r + \dots$ near $r=0$, $u(\infty) \to 0$

$e^u \sim 1 + u + O(u^2)$ so for large r $e^u \sim 1 + u$

$$u'' + \frac{1}{r} u' = u$$

Which is a Modified Bessel equation

$$r^2 u'' + r u' - (r^2 + n^2) u = 0$$

This is solved by

$$u = C_0 K_0(r) \sim C_0 \sqrt{\frac{\pi}{2r}}\, e^{-r} \quad \text{as } r \to \infty$$

Where $|C_0|$ is a Scalar Charge.

Asymptotically vortex behaves as a point like object carrying a scalar charge and dipole moment.

Vortices on Curved Surfaces

Let $\mathbb{R} \times \Sigma$ be $2+1$ dim spacetime with the metric $dt^2 - g$ where (Σ, g) is an oriented Riemannian 2-manifold.

The static Abelian Higgs model is a Hermitian complex line bundle

L with a $U(1)$ connection A over Σ.

The Higgs field ϕ is a global C^∞ section of L.

Can choose a trivialisation of L s.t. $A = A_z dz + \bar{A}_z d\bar{z}$ where $z = x_1 + ix_2$

Such that our metric on Σ is

$$g = \Omega(z, \bar{z})\, dz\, d\bar{z}$$

and $F = B\, dx_1 \wedge dx_2$ defines our magnetic field B.

Via the Bogomolny trick and starting with the energy functional

$$E = \frac{1}{2}\int_\Sigma \left[\Omega^{-1} B^2 + |D\phi|^2 + \frac{1}{4}\Omega\left(1 - |\phi|^2\right)^2\right] dx^1 \wedge dx^2$$

We find the Bogomolny equations

$$\bar{D}\phi = (\partial_{\bar{z}} - iA_{\bar{z}})\,\phi =$$

$$B = \frac{\Omega}{2}\left(1 - |\phi|^2\right)$$

Using $\phi = e^{u_2 + i\chi}$, $\bar{D}\phi$ provides A and using the second Bogomolny equation we can find (away from zeroes)

$$\Delta_0 u - \Omega(e^u - 1) = 0$$

Where $\Delta_0 = 4\partial_z \partial_{\bar{z}}$

The Bradlow bound

Assume Σ is compact. Integrating the second Bogomolny equation

$$\int B\, dx^1 \wedge dx^2 + \frac{1}{2}\int |\phi|^2 \Omega\, dx^1 \wedge dx^2 - \frac{1}{2}\int \Omega\, dx^1 \wedge dx^2 = 0$$

$$\int_\Sigma B\, dx^1 \wedge dx^2 + \frac{1}{2}\int_\Sigma |\phi|^2 \Omega\, dx^1\wedge dx^2 - \frac{1}{2}\int_\Sigma \Omega\, dx^1 \wedge dx^2 = 0$$

$$\Rightarrow \int_\Sigma F + \frac{1}{2}\int |\phi|^2 \Omega\, dx^1 \wedge dx^2 = \frac{1}{2} Area(\Sigma)$$

$$\Rightarrow 4\pi N + \int_\Sigma |\phi|^2 \Omega\, dx^1\wedge dx^2 = Area(\Sigma)$$

So $Area(\Sigma) \geq 4\pi N$ as $\int_\Sigma |\phi|^2 \Omega\, dx^1 \wedge dx^2 > 0$

An N vortex solution requires $Area(\Sigma) \geq 4\pi N$.

Maximum Principle

near $r=0$ $u = 2K\ln r + \dots$ So towards $r=0$, $u \to -\infty$

Assume Σ is compact, u must have a global maximum u_{max} on Σ.

At the maximum its Hessian must be non positive.

Using $\Delta_0 u - \Omega(e^u - 1) = 0$

$$\Delta_0 u \big|_{u_{max}} = \Omega\left(e^{u_{max}} - 1\right) \leq 0$$

which is the trace of the Hessian

$\Rightarrow u \leq 0$ So from $B = \frac{\Omega}{2}(1 - |\phi|^2) = \frac{\Omega}{2}(1 - e^u)$, $B \geq 0$ on Σ

At asymptotic region ($r = \infty$) as $u \sim 2\ln\phi$ $u \to 0$ at $r = \infty$ so

$B \to 0$ (field is expelled).

$D \to 0$ (field is expelled).

Towards $r=0$ i.e vortex core $u \to -\infty$ so $B \to \frac{\Omega}{2}$

$|\phi|$ is the order param, $|\phi| \simeq 1$ is superconducting phase,

$|\phi| \approx 0$ is the normal phase.

A vortex is a localised core where $|\phi|$ vanishes and flux is quantised, outside the core it is superconducting.

Meissner effect!

Hyperbolic vortices

If $\Omega = 1$ $(\Sigma = \mathbb{R}^2)$ then Taubes eq is non integrable.

Witten pointed out if Σ is a surface with Gaussian curvature $-\frac{1}{2}$ the Bogomolny eq's become integrable

Let $\Sigma = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

with $g = \frac{8}{(1-|z|^2)^2} dz d\bar{z} = \Omega dz d\bar{z}$

Letting $u = \sigma - \ln \Omega$ we find

$$\Delta_0 \sigma = e^{\sigma}$$

which is completely solvable.

The corresponding Higgs field is

$$\phi = \frac{1-|z|^2}{1-|f|^2} \frac{df}{dz}$$

where $|f| < 1$ and $f(z)$ is meromorphic

The vortex centres are where $f'(z)$ vanishes.

To satisfy $|\phi(\infty)| = 1$ we take f as

$$f = \prod_{k=1}^{N+1} \frac{z - c_k}{1 - \bar{c}_k z}$$

$c_k \in \mathbb{C}$, $|c_k| < 1$