Chapter 2

Sunday, 18 January 2026 15:50

The first Soliton equation is the KdV equation which models shallow water waves which are self-reinforcing. Discovered along the Edinburgh Union Canal!

KdV eq is

$$U_t - 6uu_x + u_{xxx} = 0$$

How can we motivate/derive this!?

Consider the usual wave equation

$$\Psi_{xx} - \frac{1}{v^2}\Psi_{tt} = 0$$

Where v is the waves constant velocity.

The wave eq assumes that

- This equation is t symmetric in $t \mapsto -t$ leaves the eq invariant
- amplitude oscillations are small (no non-linear terms)
- The velocity is constant (no dispersion)

From D'Alembert we know we have a general solution

$$\Psi = f(x - vt) + g(x + vt)$$

Which obey $\Psi_x + \frac{1}{v}\Psi_t = 0$

Let's introduce dispersion, Consider $\Psi = e^{i(kx - \omega(k)t)}$

Usually we have $\omega = vk$, Instead consider $\omega(k) = v(k - \beta k^3 + \ldots)$

Where we have no even terms which ensure a real dispersion relation. ($\partial_t \mapsto -i\omega$, $\partial_x \mapsto ik$ into KdV)

Let our dispersion be sufficiently small s.t $\omega(k) \sim vk - v\beta k^3$.

Assuming a wave solution

$$\Psi = e^{i(kx - v(k - \beta k^3)t)}$$

We can see

$$\Psi_x = ik\Psi, \quad \Psi_t = -iv(k-\beta k^3)\Psi, \quad \beta\Psi_{xxx} = -i\beta k^3 \Psi$$

So...
$$\Psi_x + \frac{1}{v}\Psi_t + \beta\Psi_{xxx} = 0$$

If we identify $\rho = \frac{1}{v}\Psi$, $j = \Psi + \beta\Psi_{xx}$

we can state this as a continuity equation

$$\rho_t + j_x = 0$$

We can introduce a nonlinear term as

$$j = \Psi + \beta\Psi_{xx} + \frac{\alpha}{2}\Psi^2$$

So our continuity equation is finally

$$\frac{1}{v}\Psi_t + \Psi_x + \beta\Psi_{xxx} + \alpha\Psi\Psi_x = 0$$

The Ψ^2 term is the relaxation of the low amplitude assumption

It's the lowest order correction as we already have a Ψ term.

If we now change variables to $x \mapsto x - vt$ and rescale Ψ we find

$$u_t - 6uu_x + u_{xxx} = 0$$

The simplest solution is

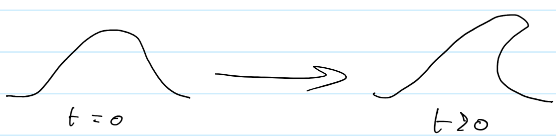
$$u(x,t) = \frac{-2\chi^2}{\cosh^2\chi(x - 4\chi^2 t - \phi_0)}$$

Nonlinear equation! No superposition principle.

The self-reinforcing nature comes from u_{xxx} in that

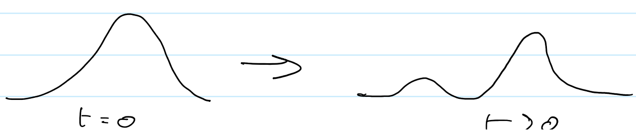
$$u_t - 6uu_x = 0$$

exhibits waves like

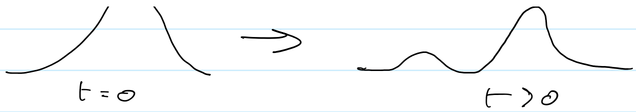


(breaking waves)

and $-6uu_x$ contributes in that $u_t + u_{xxx} = 0$ has waves like



(dispersive)



(dispersive)

2.12) Backlund transforms for the Sine-Gordon equation

The Sine-Gordon equation is given by

$$\phi_{xx} - \phi_{tt} = \sin\phi$$

The Backlund transformation is given by $\tau = \frac{x+t}{2}$, $\rho = \frac{x-t}{2}$

$x = \rho + \tau, \quad t = \tau - \rho$

$$\partial_\tau = \frac{\partial x}{\partial \tau}\partial_x + \frac{\partial t}{\partial \tau}\partial_t = \partial_x + \partial_t$$

$$\partial_\rho = \frac{\partial x}{\partial \rho}\partial_x + \frac{\partial t}{\partial \rho}\partial_t = \partial_x - \partial_t$$

so $\partial_x = \frac{1}{2}(\partial_\tau + \partial_\rho)$, $\partial_t = \frac{1}{2}(\partial_\tau - \partial_\rho)$

$$\partial_x^2 = \frac{1}{4}(\partial_\tau^2 + 2\partial_\tau\partial_\rho + \partial_\rho^2), \quad \partial_t^2 = \frac{1}{4}(\partial_\tau^2 - 2\partial_\tau\partial_\rho + \partial_\rho^2)$$

$$\partial_x^2 - \partial_t^2 = \frac{1}{4}(4\,\partial_\tau\partial_\rho) = \partial_\tau\partial_\rho$$

so $\phi_{xx} - \phi_{tt} = \sin\phi \mapsto \partial_\tau\partial_\rho\phi = \sin\phi$

Define the Backlund relations

$$\partial_\rho(\phi_1 - \phi_0) = 2b\sin\left(\frac{\phi_1 + \phi_0}{2}\right), \quad \partial_\tau(\phi_1 + \phi_0) = 2b^{-1}\sin\left(\frac{\phi_1 - \phi_0}{2}\right)$$

Then

$$\partial_\tau\partial_\rho(\phi_1 - \phi_0) = 2b\,\partial_\tau\sin\left(\frac{\phi_1 + \phi_0}{2}\right) = 2\cos\left(\frac{\phi_1 + \phi_0}{2}\right)\sin\left(\frac{\phi_1 - \phi_0}{2}\right)$$

$$= \sin\phi_1 - \sin\phi_0$$

This means ϕ_1 solves S.G if ϕ_0 also does.

Given some ϕ_0 we can solve the first order Backlund relations for ϕ_1 to generate new solutions. Using $\phi_0 = 0$ we find

$$\phi_1(x,t) = 4\arctan\left(e^{\gamma(x - vt) - x_0}\right)$$

where $\gamma = (1 - v^2)^{-\frac{1}{2}}$

This is a kink solution, we can define a topological charge

$$N = \frac{1}{2\pi}\int_{\mathbb{R}} d\phi = \frac{1}{2\pi}\left[\phi(-\infty,t) - \phi(\infty,t)\right]$$

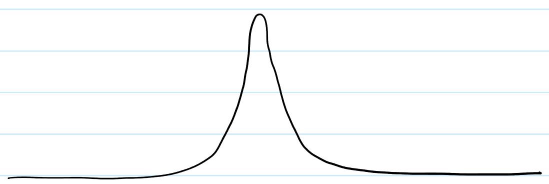
ϕ is connecting vacua of the potential $V(\phi) = (1-\cos\phi)$ i.e $\cos\phi = 1$ which is for $\phi = 2n\pi$. ϕ in a way is then living on S^1 so $N = \frac{1}{2\pi}\int d\phi$ is the degree of ϕ or how many times it wraps around S^1.

We have a conserved energy

$$E = \int\left[\frac{1}{2}\left(\phi_t^2 + \phi_x^2\right) + \left(1-\cos\phi\right)\right]dx$$

The Sine-Gordon equation is the variational equation of the above functional. The kink looks like

While its energy density looks like



The kink is stable in that it would take infinite energy to deform it to $\phi = 0$.

We can have a soliton-antisoliton pair

$$\phi(x,t) = 4\arctan\left(\frac{v\cosh(\gamma x)}{\sinh(\gamma v t)}\right)$$

Which is a non-trivial $N=0$ solution.

2.2) Inverse Scattering transform for KdV

Consider the 1D Schrodinger equation in $\frac{\hbar^2}{2m} = 1$ units

$$\left[-\frac{d^2}{dx^2} + u(x)\right]\Psi = E\Psi$$

Functions which obey $\langle \Psi . \Psi \rangle < \infty$ are called bound states otherwise these are called scattering states. An example is e^{-ix}.

In scattering theory, consider a beam of free particles incident from $+\infty$. Some will be deflected by the potential and some transmitted. We may also have bound states with discrete energy levels.

Gelfand-Levitan-Marchenko (GLM) theory shows given

Energy levels E_n

Transmission probability T

Reflection probability R

We can find the potential u. Given $u_0(x)$ we can find the $t=0$ scattering data. If $u(x,t)$ solves KdV with $u(x,0) = u_0(x)$ then $(E(t), T(t), R(t))$ obey linear ODE's determining their time evolution. $u(x,t)$ is recovered by solving a linear integral equation.

$$\begin{tikzcd}
\begin{array}{c} u_t - 6uu_x + u_{xxx} = 0 \\ u(x,0) = u_0(x) \end{array} \arrow[r, "S.E"] \arrow[d, dashed] & \text{Scattering @ } t=0 \arrow[d, "\begin{array}{c}\text{KdV} \\ \text{Lax pair}\end{array}"] \\
u(x,t) & \begin{array}{c}\text{Scattering} \\ \text{@ } t>0\end{array} \arrow[l, "\text{GLM eq}"']
\end{tikzcd}$$

2.2.1) Direct Scattering

Let $E = k^2$ and L denote Schrodinger operator $L = -\frac{d^2}{dx^2} + u(x)$

$$Lf = k^2 f$$

Let $E = -k^2$ and L denote Schrodinger operator $L = -\frac{d^2}{dx^2} + u(x)$

$$Lf = k^2 f$$

Consider the potentials which obey $|u(x)| \to 0$ as $x \to \pm\infty$ and

$$\int_{\mathbb{R}} (1 + |x|)\,|u(x)|\,dx < \infty$$

Which guarantees a finite number of discrete energy levels

At $x = \pm\infty$ the system behaves as a free particle

$$-f_{xx} = k^2 f$$

with $f = C_1 e^{ikx} + C_2 e^{-ikx}$.

For each non zero k, the set of solutions to $Lf = k^2 f$ form a 2D vector space G_k. Since $u(x)$ is real if f satisfies $Lf = k^2 f$, then $L\bar{f} = k^2 \bar{f}$.

Consider $(\psi, \bar{\psi})$, $(\phi, \bar{\phi})$ to be 2 bases of G_k determined by

$$\psi(x,k) \simeq e^{-ikx}, \quad \bar{\psi}(x,k) \simeq e^{ikx} \quad \text{as } x \to \infty$$

and

$$\phi(x,k) \simeq e^{-ikx}, \quad \tilde{\phi}(x,k) \simeq e^{ikx} \quad \text{as } x \to -\infty$$

Expand solution in first basis

$$\phi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k)$$

So if $a \neq 0$ we have

$$\frac{\phi(x,k)}{a(k)} = \begin{cases} \dfrac{e^{-ikx}}{a(k)} & x \to -\infty \\ e^{-ikx} + \dfrac{b(k)}{a(k)} e^{ikx} & \text{for } x \to \infty \end{cases}$$

Consider particle incident from ∞ with wave func e^{-ikx}.

The transmission and reflection coefficent are

The transmission and reflection coefficent are

$$t(k) = \frac{1}{a(k)}, \quad r(k) = \frac{b(k)}{a(k)}$$

which satisfy $|t(k)|^2 + |r(k)|^2 = 1$

2.2.2) Properties of scattering data

Let $k \in \mathbb{C}$. From scattering theory it is known

- $a(k)$ is holomorphic in the upper half plane
- $\{\mathrm{Im}(k) \geq 0, |k| \to \infty\} \to |a(k)| \to 1$
- Zeroes of $a(k)$ in upper half plane lie on imaginary axis. The number of zeroes is finite if $\int (1+|x|)|u(x)|\,dx < \infty$

So $a(i\chi_1) = \cdots = a(i\chi_n) = 0$ where $\chi_1 > \dots > \chi_n > 0$

- Since $\phi(x, i\chi_n) = \begin{cases} e^{-i(i\chi_n)x} & x \to -\infty \\ a(i\chi_n)e^{-i(i\chi_n)x} + b(i\chi_n)e^{i(i\chi_n)} & \text{for } x \to \infty \end{cases}$

So the asymptotics of ϕ at these zeroes are

$$\phi(x, i\chi_n) = \begin{cases} e^{\chi_n x} & x \to -\infty \\ b_n e^{-\chi_n} & x \to \infty \end{cases}$$

It can be shown $b_n = (-1)^n |b_n|$ and $ia'(i\chi_n)$ has same sign as b_n.

Have $\left[-\frac{d^2}{dx^2} + u(x)\right]\phi(x, i\chi_n) = -\chi_n^2 \phi(x, i\chi_n)$

So ϕ is square integrable with $E = -\chi_n^2$

2.2.3) Inverse scattering

To recover $u(x)$ from scattering data $r(k), \{\chi_1, \dots, \chi_N\}$

Where $E_n = -\chi_n^2$ and

$$\phi(x, i\chi_n) = \begin{cases} e^{\chi_n x} & x \to -\infty \\ b_n e^{-\chi_n x} & x \to \infty \end{cases}$$

$$\phi(x, i\chi_n) = \begin{cases} e^{\cdots} & x \to -\infty \\ b_n e^{-\chi_n x} & x \to \infty \end{cases}$$

To set up the inverse scattering transform first set

$$F(x) = \sum_n \frac{b_n e^{-\chi_n x}}{i a'(i\chi_n)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk$$

Then solve the GLM equation for K

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,z) F(z+y) dz = 0$$

then $\quad u(x) = -2 \frac{d}{dx} K(x,x)$

is the schrodinger eq potential.

If the t dependence of the scattering data is known, $K(x,y,t)$ and $u(x,t)$ pick up t dependence.

2.2.4) Lax formulation

If $u(x)$ depends on t, the eigenvalues of the S.E will depend on t.

Proposition 2.2.1)

If there exists a differential operator A s.t

$$\dot{L} = [L, A]$$

where $\quad L = -\frac{d^2}{dx^2} + u(x,t)$

the spectrum of L is t independent.

Proof: $\quad Lf = Ef$

$$L_t f + L f_t = E_t f + E f_t$$

Noting $ALf = EAf$

$$(L-E)(f_t+Af)=E_t f$$

Taking the inner product of this with f and use L is self adjoint

$$E_t\|f\|^2=\langle f,(L-E)(f_t+Af)\rangle=\langle \underbrace{(L-E)f}_{0}, f_t+Af\rangle=0$$

So $E_t=0$

This is applicable to KdV in that

$$L=-\frac{d^2}{dx^2}+u(x,t),\ A=4\frac{d^3}{dx^3}-3\left(u\frac{d}{dx}+\frac{d}{dx}u\right)$$

is equivalent to KdV. $[L,A]$ is multiplication by $6uu_{x}-u_{xxx}$ and $\dot{L}=u_t$

2.2.5) Evolution of scattering data.

Assume $u(x,t)$ from S.E satisfies KdV. Let f satisfy $Lf=k^2f$ defined by

$$f=\phi(x,k)\mapsto e^{-ikx}\quad \text{as } x\to-\infty.$$

Due to $(L-E)(f_t+Af)=\overbrace{E_t f}^{0}$, if $Lf=k^2f$, (f_t+Af) is an eigenvector as $u(x)\to 0$ as $|x|\to\infty$

$$\dot{\phi}+A\phi\longmapsto 4\frac{d^3}{dx^3}e^{-ikx}=4ik^3e^{-ikx}\quad \text{as } x\to-\infty$$

So $4ik^3\phi(x,k)$ and $\dot{\phi}+A\phi$ are eigenfunctions of L with the same asymptotics hence they must be equal. We get the ODE

$$\dot{\phi}+A\phi=4ik^3\phi$$

Recall $\phi(x,k)=a(k,t)e^{-ikx}+b(k,t)e^{ikx}$ as $x\to\infty$

$$\Rightarrow \quad \dot{a}\, e^{-ikx} + \dot{b}\, e^{ikx} = 8ik^3 b e^{ikx}$$

So $\dot{a} = 0$, $\quad \dot{b} = 8ik^3 b$, $\; a(k,t) = a(k,0)$, $\; b(k,t) = b(k,0)\, e^{8ik^3 t}$

We can then find $r(k,t) = r(k,0)\, e^{8ik^3 t}$, $\; \chi_n(t) = \chi_n(0)$

$b_n(t) = b_n(0)\, e^{8\chi_n^3 t}$, $\; a_n(t) = 0$, $\; \beta_n(t) = \beta_n(0)\, e^{8\chi_n^3 t}$.

2.3.1) One Soliton Solution

Recall
$$F(x) = \sum_{n=1}^{N} \frac{b_n e^{-\chi_n x}}{i a'(i\chi_n)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)\, e^{ikx}\, dk$$

If $r = 0$ initially it remains 0. For $r = 0$ we can explicitly solve this

For $N = 1$

$$F(x) = \frac{b_n(t) e^{-\chi x}}{i a'(i\chi)} = \beta(t)\, e^{-\chi x} = \beta(0)\, e^{8\chi^3 t - \chi x}$$

Treat the t dependence as a parameter.

GLM eq becomes

$$K(x,y) + \beta e^{-\chi(x+y)} + \int_x^{\infty} K(x,z)\, \beta\, e^{-\chi(z+y)}\, dz = 0$$

Search for $K(x,y) = K(x)\, e^{-\chi y}$ $\Rightarrow$

$$K(x) + \beta e^{-\chi x} + K(x)\, \beta \int_x^{\infty} e^{-2\chi z}\, dz = 0$$

So
$$K(x) = -\frac{\beta e^{-\chi x}}{1 + \frac{\beta}{2\chi} e^{-2\chi x}}$$

so
$$K(x,y) = -\frac{\beta e^{-\chi(x+y)}}{1 + \frac{\beta}{2\chi} e^{-2\chi x}}$$

Since $U(x,t) = -2\frac{\partial}{\partial x} K(x,x) = -\frac{4\beta\chi e^{-2\chi x}}{(1+\frac{\beta}{2\chi}e^{-2\chi x})^2} =$

$$= -\frac{8\chi^2}{(\tilde{\beta}^{-1}e^{\chi x} + \tilde{\beta}e^{-\chi x})^2}$$

$$= -\frac{2\chi^2}{\cosh(\chi(x-4\chi^2 t-\phi_0))^2} \qquad \text{Where } \phi_0 = \frac{1}{2\chi}\ln\left(\frac{\beta_0}{2\chi}\right)$$

The energy of the corresponding S.E solution determines the amplitude and velocity of the soliton. The soliton is of form

$$u = u(x-4\chi^2 t)$$

Which is a wave travelling with $v = 4\chi^2$ to the right with phase ϕ_0.