The Brauer-Manin Obstruction (Lewis)

Brauer groups of fields (§1.5)

Def An Azumaya algebra over a field k is a k-algebra A with $A \otimes_k k_s$ isomorphic to the matrix algebra $M_n(k_s)$ for some $n \geq 1$

Prop The following are equivalent.

i) A is an Azumaya algebra over k

ii) A is a k-algebra which is finite dimensional, central ($Z(A) = k$) and simple (only 2-sided ideals are 0 and A).

iii) There is a finite separable extension L/k with $A \otimes_k L \cong M_n(L)$ for some $n \geq 1$.

iv) There is an extension L/k with $A \otimes_k L \cong M_n(L)$ for some $n \geq 1$.

v) $A \cong M_r(D)$ for some finite dimensional division algebra D over k, for some $r \geq 1$.

~~Prop~~ Write Az_k for the category of Azumaya algebras over K

Prop

i) if $A \in Az_k$ then $A^{opp} \in Az_k$

ii) if $A, B \in Az_k$ then $A \otimes_k B \in Az_k$

iii) if $A \in Az_k$ and L/k an extension then $A \otimes_k L \in Az_L$.

Def $A \in Az_k$ is split if $A \cong M_n(k)$.

<u>Def</u> $A, B \in Az_k$ are similar $(A \sim B)$ if one of the following equivalent conditions holds.

i) $\exists$ division algebra $D \in Az_k$ and $m, n \geq 1$ with $A \cong M_m(D)$, $B \cong M_n(D)$

ii) $\exists\, m, n \geq 1$ with $M_m(A) \cong M_n(B)$

Define $\mathrm{Br}(K)$ as the set of equivalence classes $Az_k/\sim$. This is a group (the Brauer group) with multiplication $\otimes_k$, inverses $A \mapsto A^{opp}$, and identity given by k (and all other split Azumaya algebras).

<u>Theorem</u> This is canonically isomorphic to our previous definition of $\mathrm{Br}\, k$ as $H^2(k, \mathbb{G}_m)$.

Let L/k be a degree n cyclic field extension and $a \in k^\times$, $\sigma \in \mathrm{Gal}(L/k)^*$. Let $L[x]_\sigma$ be the twisted polynomial ring, which has additive group $L[x]$ and multiplication defined by $x\ell := ({}^\sigma \ell)x$

<u>Prop</u> $(a, \sigma) \in Az_k$.

Take a fixed L, σ and write $G = \mathrm{Gal}(L/k) \cong \mathbb{Z}/n\mathbb{Z}$. Tate cohomology gives that the following commutes.

$$\begin{tikzcd}
k^\times \arrow[r, two heads] \arrow[ddrr, "a \mapsto (a,\sigma)"'] & k^\times/N_{L/k}(L^\times) = \hat{H}^0(G, L^\times) \arrow[r, "\sim"] & H^2(G, L^\times) \arrow[d, "\cong"] \\
& & \ker(\mathrm{Br}\, k \to \mathrm{Br}\, L) \arrow[d, hook] \\
& & \mathrm{Br}\, k
\end{tikzcd}$$

---

* σ must be a generator of $\mathrm{Gal}(L/k)$.

Prop (a, σ) is split iff $a \in N_{L/k}(L^\times)$

proof compare kernels in the diagram.

Def Take $a, b \in k^\times$. Let $L = k[t]/\langle t^2 - b\rangle$, and $\sigma \in Gal(L/k)$ the element sending t to $-t$. Write (a,b) for (a,σ) in this case.

Prop In the Brauer group,
$(a,b) + (c,b) = (ac, b)$, and therefore
$(a,b) - (c,b) = (a/c, b)$.

corollary every (a,b) is 2-torsion in $Br\,k$

proof $2(a,b) = (a^2, b)$, and a^2 is the L/k-norm of a.

The Brauer-Manin Obstruction (for real now)

Hope: Hasse Principle
Let X be a variety over a global field k, which admits a point over every completion of k. Then X admits a point over k.

This is not true in general: e.g.
$(x^2-2)(x^2-17)(x^2-34)=0$ has points in $\mathbb{R}$ and all $\mathbb{Q}_p$, not $\mathbb{Q}$.

Functorial obstructions to the Hasse Principle

Let k be a global field. Recall the adèle ring
$$A_k := \left\{ (x_v) \in \prod_v k_v \mid \text{each } x_v \in \mathcal{O}_{k_v} \text{ for all but finitely many } v \right\}.$$

Let $F : \mathrm{Sch}/k^{op} \to \mathrm{Set}$ be a functor. For a k-algebra L write $F(L) := F(\mathrm{spec}\, L)$.
Let X be a variety over k and L a k-algebra. Then each $x \in X(L)$ defines a morphism $\mathrm{Spec}\, L \to X$, which induces $F(X) \to F(L)$.

Def For $a \in F(X)$, let $\mathrm{ev}_a : X(L) \to F(L)$ be the map sending $x \in X(L)$ to the image of a under $F(X) \to F(L)$.

Then
$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d, "\mathrm{ev}_a"'] & X(A_k) \arrow[d, "\mathrm{ev}_a"] \\ F(k) \arrow[r] & F(A_k) \end{tikzcd}$$
commutes.

Def $X(\mathbb{A}_k)^a := \{x \in X(\mathbb{A}_k) \mid ev_a(x) \in \operatorname{im}(F(k) \to F(\mathbb{A}_k))\}$

The commutativity shows that $X(k) \subseteq X(\mathbb{A}_k)^a$, so we see $X(k) \subseteq X(\mathbb{A}_k)^F := \bigcap_{a \in F(X)} X(\mathbb{A}_k)^a$.

$\boxed{\text{Prop } X \mapsto X(\mathbb{A}_k)^F \text{ is functorial in } X}$

Note $X(\mathbb{A}_k) \subseteq X(\prod k_v) = \prod X(k_v)$, with equality if X is proper. Repeat the above construction with $X(\prod k_v) = \prod X(k_v)$ in place of $X(\mathbb{A}_k)$ and $\prod F(k_v)$ in place of $F(\mathbb{A}_k)$ to get $X(\prod k_v)^F \supseteq X(k)$.

Def There is an 'F-obstruction to the Hasse principal' if $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A})^F = \emptyset$.
There is an 'F-obstruction to weak approximation' if $X(\prod k_v) \neq \emptyset$ but $X(\prod k_v)^F = \emptyset$.

Examples
- Let G be an affine algebraic group over k, and take $F = H^1(-, G)$
- Take $F = \mathrm{Br}$ to get the Brauer set $X(\mathbb{A}_k)^{\mathrm{Br}}$.

Brauer - Manin obstruction

Recall that for a non-archimedean local field K there is an isomorphism $\mathrm{Br}\, K \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ coming from the invariant map in class field theory.
Using that $\mathrm{Br}\, \mathbb{R} \cong \mathbb{Z}/2\mathbb{Z} \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$
and $\mathrm{Br}\, \mathbb{C} \cong 0 \cong \mathbb{Z}/\mathbb{Z}$,
we see that every local field K admits an injection $\mathrm{inv} : \mathrm{Br}\, K \to \mathbb{Q}/\mathbb{Z}$.

~~Fix a global field k, and take~~

Fix a global field k and a variety X/k.
Take $a \in \operatorname{Br} X$. For each completion k_v of k, write $\mathrm{inv}_v : \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$.

Prop if $(x_v) \in X(\mathbb{A}_k)$ then $\mathrm{ev}_a(x_v) = 0$ for almost all v.

Therefore a determines a map $(a, -) : X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ defined by $(a, (x_v)) = \sum_v \mathrm{inv}_v(\mathrm{ev}_a(x_v))$

Prop If $x \in X(k) \subseteq X(\mathbb{A}_k)$ then $(a, x) = 0$

proof
$$\begin{tikzcd}
 & X(k) \arrow[r, hook] \arrow[d, "\mathrm{ev}_a"'] \arrow[dr, phantom, "\circlearrowleft"] & X(\mathbb{A}_k) \arrow[d, "\mathrm{ev}_a"] & & \\
0 \arrow[r] & \operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$
.

~~Theorem The natural homomorphism~~
Theorem (Česnavičius 2015)
The natural homomorphism $\operatorname{Br} \mathbb{A}_k \to \bigoplus_v \operatorname{Br} k_v$ is an isomorphism

corollary $X(\mathbb{A}_k)^a = \{(x_v) \in X(\mathbb{A}_k) \mid (a, (x_v)) = 0\}$

Conjecture If X is a nice variety over a global field k then the Brauer-Manin obstruction is the only obstruction to the Hasse principle, i.e. if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ then $X(k) \neq \emptyset$.

Iskovskikh's conic bundle

Let E be the k-vector space spanned by ~~x_1, x_2, x_3~~ x, y, z. Then $\mathrm{Sym}\, E = k[x, y, z]$ and $\mathrm{Sym}^2 E$ is given by the degree 2 polynomials, so a conic in $\mathbb{P}^2$ is the zero locus in $\mathrm{Proj}\, \mathrm{Sym}\, E$ of an element $s \in \mathrm{Sym}^2 E$.

Def Let B be a k-scheme and let $\mathcal{E}$ be a rank 3 vector bundle on B. A conic bundle over B is the zero locus in $\mathrm{Proj}\, \mathrm{Sym}\, \mathcal{E}$ of some nowhere-vanishing $s \in \Gamma(B, \mathrm{Sym}^2 \mathcal{E})$.

A châtelet surface is a conic bundle in the case $B = \mathbb{P}^1$, $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$, and $s = 1 - a - F(w, x)$ for some $a \in k^\times$ and $F(w, x) \in \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ is a separable homogeneous degree 4 polynomial in the homogeneous coordinates w, x on $\mathbb{P}^1$.

This is a surface X over k that contains the affine surface $y^2 - az^2 = f(x) := F(1, x)$ as an open subscheme. It comes with a map $X \longrightarrow \mathbb{P}^1$, whose fibres are all smooth conics, except above the 4 zeros of F, whee the fibres are given by a pair of intersecting lines.

Iskovskikh's surface X is the Châtelet surface corresponding to $a = -1$, $f(x) = (3 - x^2)(x^2 - 2)$ over $k = \mathbb{Q}$.
Let $K = k(X)$.
For any $f, g \in K^\times$, write (f, g) for the quaternion algebra over K which is generated

~~by i, j, k such that $i^2 = a$, $j^2 = b$, $ij = k$, $ji = -k$~~
~~This is central simple, so represents a class in~~
~~$Br K$. In fact, $(a,b) \in Br K[2]$.~~

by i, j, k such that $i^2 = f$, $j^2 = g$, $ij = k$, $ji = -k$.
Fact (f, g) represents a class in $Br K[2]$.

Define $a = (3 - x^2, -1) \in Br K$. Recall there is a natural inclusion $Br X \hookrightarrow Br K$.

Prop $a \in Br X$.

proof Recall the exact sequence

$$0 \to Br X \longrightarrow Br k(X) \xrightarrow{res} \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbb{Q}/\mathbb{Z}).$$

~~It suffices to sh~~ where $X^{(1)} = \{$codimension 1 points on $X\}$ is in bijection with the set of integral divisors. It then suffices to show that a has no residue on any integral divisor of X. So it suffices to find a Zariski open covering $\{U_i\}$ of X such that a extends to an element of $Br U_i$ for each i.

Let $b = (x^2 - 2, -1)$ and $c = (3/x^2 - 1, -1)$ in $Br K$
~~Then $a + b = (y^2 + z^2, -1)$, which is zero in the Brauer group.~~
~~Then $a + b = ((3 - x^2)(x^2 - 2), (-1)$~~
Then $a + b = ((3 - x^2)(x^2 - 2), -1) = (y^2 + z^2, -1)$, which is 0 in the Brauer group, and
$a - c = a + \left(\frac{1}{3/x^2 - 1}, -1\right) = (x^2, -1) = 2(x, -1) = 0$

So since they are 2-torsion we have $a = b = c$.

~~Proposition if $(x_v) \in X(\mathbb{A}_k)$ then ev_a~~

$a = (3-x^2, -1)$ represents a quaternion algebra wherever $3-x^2$ doesn't have a zero or pole. Writing P_{3-x^2} for the closed point of $\mathbb{P}^1_{\mathbb{Q}}$ given by $3-x^2=0$ and X_p for the fibre above p, we see that a defines an element of $\operatorname{Br} U_a$, with

$U_a := X - X_\infty - X_{P_{3-x^2}}$

Since $a=b=c$, it also defines elements of $\operatorname{Br} U_b$ and $\operatorname{Br} U_c$, where

$U_b := X - X_\infty - P_{x^2-2}$

$U_c := X - X_0 - P_{3-x^2}$

And $U_a \cup U_b \cup U_c = X$

<u>Prop</u> $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ but $X(\mathbb{A}_{\mathbb{Q}})^a = \emptyset$, so there is a Brauer-Manin obstruction and $X(\mathbb{Q}) = \emptyset$.

<u>Proof</u> X contains the ~~conic~~ surface $y^2+z^2 = (3-x^2)(x^2-2)$, which clearly contains real solutions, and Hensel's lemma can be used to show it admits p-adic solutions, so $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$.

Suppose $P \in X(\mathbb{Q}_p)$ for p prime or ∞. write $x := x(P) \in \mathbb{Q}_p \cup \{\infty\}$.

<u>Case 1</u>: $p \notin \{2, \infty\}$. ~~If |x|~~

If $|x|_p < 0$ or $x = \infty$, then $3/x^2 - 1 \in \mathbb{Z}_p^\times$.

If $|x|_p \geq 0$ then at least one of $3-x^2$ and

x^2-2 is in $\mathbb{Z}_p^\times$, since their sum is 1.
So in either case $ev_a(P) = (u_1, u_2)$ for $u_i \in \mathbb{Z}_p^\times$, so $ev_a(P) \in \mathrm{Br}\,\mathbb{Z}_p$. (this uses $p \neq 2$)
But recall the Brauer group of the valuation ring of a nonarchimedean local field is 0,
so $\mathrm{inv}_p\, ev_a P = 0$

<u>case 2</u> $p = \infty$. Changing to coordinates $(X, Y, Z) = (\frac{1}{x}, \frac{y}{x^2}, \frac{z}{x^2})$,
The equation $y^2 + z^2 = (3 - x^2)(x^2 - 2)$
becomes $Y^2 + Z^2 = (3X^2 - 1)(1 - X^2)$
so at $x = \infty$ one has $Y^2 + Z^2 = -1$.
This has no real solutions, so every $P \in X(\mathbb{R})$ has $x \neq \infty$. Then $x^2 < 3$ or $x^2 > 2$, so one of $3 - x^2$, $x^2 - 2$ lies in $\mathbb{R}_{>0} = N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$,
so $\mathrm{inv}_\infty\, ev_a P = 0$

<u>case 3</u> $p = 2$. Then

$|x|_2 > 0 \Rightarrow 3 - x^2 \equiv -1 \pmod 4$
$|x|_2 = 0 \Rightarrow x^2 - 2 \equiv -1 \pmod 4$
$|x|_2 < 0 \Rightarrow 3/x^2 - 1 \equiv -1 \pmod 4$

and $f \equiv -1 \bmod 4 \Rightarrow \nexists\, a, b \in \mathbb{Q}_2$ with $a^2 + b^2 = f$,
so f is not a norm from $\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2$.
So a represents an order 2 element of $\mathrm{Br}\,X$,
so $\mathrm{inv}_2\, ev_a P = 1/2$

so if $(P_p) \in X(\mathbb{A}_\mathbb{Q})$ then $(a, (P_p)) = 1/2 \neq 0$

So $X(\mathbb{A}_\mathbb{Q})^a = \emptyset$.