Integrability of (A)SDYM and Twistor Theory

AGQ Reading Group: Istantons, Solitons, and Twistors

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1 Introduction

These notes were presented in two two-hour talks as part of the AGQ Reading Group: Instantons, Solitons, and Twistors and are closely based on Chapter 7 and Appendix B of *Solitons, Instantons, and Twistors (Second Edition)* by Maciej Dunajski [1] and *Lectures on Twistor Theory* by Tim Adamo [2].

1.1 Objectives and Structure

The aim of these lectures is introducing elements of Twistor Theory, highlighting their relation with the integrability of certain Self-Dual (SD) or Anti-Self-Dual (ASD) sectors of solutions of the four-dimensional Yang-Mills (YM) equations and the Yang-Mills Instantons. The structure of these notes is as follow: in Section 2, we review how (A)SDYM equations arise, their link with YM instantons, and we study their integrability thorugh the Lax Pair; in Section 3, we introduce the Twistor Correspondence, the definition of Twistor Space, its conformal properties, and its complex structure; in Section 4 we introduce the Ward Correspondence – the main result of this lecture – which relates (A)SDYM instantons with certain holomorphic vector bundles over Twistor Space; in Section 5, we introduce the notion of holomorphic frames, we give a strategy to reconstruct the gauge fields solutions, and work out a few explicit examples.

1.2 Notation

We briefly summarise our notation and conventions.

Spinor Formalism: Let us consider complexified Minkowski $\mathbb{M}_{\mathbb{C}} = \left(\mathbb{C}^4, \eta^{ab}\right)$ with complex coordinates $x^a \in \mathbb{C}$ and holomorphically-extended metric $\eta^{ab} = \text{diag}(+,-,-,-)^{ab}$, and denote $SO(4,\mathbb{C})$-indices with $a = 0,1,2,3$, negative chirality $SL(2,\mathbb{C})$-indices with $\alpha = 0,1$ and positive chirality $SL(2,\mathbb{C})$-indices with $\dot\alpha = 0,1$.

The Spin-group of the spacetime is $\text{Spin}\left(\mathbb{M}_{\mathbb{C}}\right) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, double cover of $SO(4,\mathbb{C}) \cong \frac{SL(2,\mathbb{C})\times SL(2,\mathbb{C})}{\pm\{\mathbb{1},\mathbb{1}\}}$. This isomorphism is carried out through Pauli matrices $\sigma_a^{\alpha\dot\alpha}$:

$$v^{\alpha\dot\alpha} = \frac{1}{2}\sigma_a^{\alpha\dot\alpha} v^a = \begin{pmatrix} v^0+v^3 & v^1+iv^2 \\ v^1-iv^2 & v^0-v^3 \end{pmatrix} \tag{1.1}$$

The $SL(2,\mathbb{C})$-invariant, antisymmetric tensor

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{\dot\alpha\dot\beta}. \tag{1.2}$$

defines a skew-symmetric inner product on spinorial objects in some representation of each $SL(2,\mathbb{C})$

copy and can be used to lower and higher $SL(2,\mathbb{C})$-indices:

$$k^{\alpha} w_{\alpha} = k^{\alpha} w^{\beta} \varepsilon_{\beta\alpha} = \langle k w \rangle\,, \tag{1.3}$$

$$\tilde{k}^{\dot\alpha} \tilde{w}_{\dot\alpha} = \tilde{k}^{\dot\alpha} \tilde{w}^{\dot\beta} \varepsilon_{\dot\beta\dot\alpha} = \left[\tilde{k}\tilde{w}\right]. \tag{1.4}$$

Combining these relations, we have $v^2 = v^a v_a = 2\det\left(v^{\alpha\dot\alpha}\right)$. This implies that every null $SO(4,\mathbb{C})$-vector can be written as $k = \lambda^{\alpha}\tilde{\lambda}^{\dot\alpha}$.

The line element can be written as $ds^2 = \eta_{ab} dx^a dx^b = \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} dx^{\alpha\dot\alpha} dx^{\beta\dot\beta}$. Note that the concept of signature loses meaning for complex coordinates.

We introduce a second set of complex coordinates on $\mathbb{M}_{\mathbb{C}}$, $Z^i = (z, \tilde{z}, w, \tilde{w})$ with $i = 1,2,3,4$ and referred to as *half-lightcone coordinates*, equipped with metric $ds^2 = 2(dz\, d\tilde{z} - dw\, d\tilde{w})$. The change of variable is given by

$$z = \frac{x^0 + x^3}{\sqrt{2}},\ \tilde{z} = \frac{x^0 - x^3}{\sqrt{2}},\ w = -\frac{v^1 - iv^2}{\sqrt{2}},\ \text{and } \tilde{w} = -\frac{v^1 + iv^2}{\sqrt{2}}. \tag{1.5}$$

Hodge Operator and (Anti-)Self-Duality: On a four-dimensional real manifold $(\mathcal{M}, g)$, we define the Hodge Operator action on 2-tensors ω as

$$(\star\omega)^{ab} = \frac{1}{2}\varepsilon^{ab}_{cd}\omega^{cd}, \qquad (\star\omega)_{ab} = \frac{1}{2}\varepsilon^{cd}_{ab}\omega_{cd}. \tag{1.6}$$

where $\tilde{\varepsilon}_{abcd}$ is the fully-antisymmetric Levi-Civita symbol and $\varepsilon_{abcd} = \sqrt{|g|}\tilde{\varepsilon}_{abcd}$ is the fully-antisymmetric Levi-Civita tensor. In Lorentzian signature, $\star^2 = -1$, therefore eigenvalues for the Hodage Operator $\star$ are $\pm i$, while in Euclidean signature, $\star^2 = 1$, therefore eigenvalues are ± 1.

We extend this definition to spinorial notation. We decompose $\omega_{ab} = \sigma^{\alpha\dot\alpha}_{[a}\sigma^{\beta\dot\beta}_{b]}\left(\varepsilon_{\alpha\beta}\omega_{(\dot\alpha\dot\beta)} + \varepsilon_{\dot\alpha\dot\beta}\omega_{(\alpha\beta)}\right)$, which yields

$$(\star\omega)_{ab} = \frac{1}{2}\varepsilon^{cd}_{ab}\sigma^{\alpha\dot\alpha}_{[c}\sigma^{\beta\dot\beta}_{d]}\left(\varepsilon_{\alpha\beta}\omega_{(\dot\alpha\dot\beta)} + \varepsilon_{\dot\alpha\dot\beta}\omega_{(\alpha\beta)}\right) = i\left(\varepsilon_{\alpha\beta}\omega_{(\dot\alpha\dot\beta)} - \varepsilon_{\dot\alpha\dot\beta}\omega_{(\alpha\beta)}\right). \tag{1.7}$$

We refer to $\omega_{(\dot\alpha\dot\beta)}$ as the Self-Dual component of $\omega_{\alpha\dot\alpha\beta\dot\beta}$, which has eigenvalue $+i$, while we refer to $\omega_{(\alpha\beta)}$ as the Anti-Self-Dual component of $\omega_{\alpha\dot\alpha\beta\dot\beta}$, which has eigenvalue $-i$.

2 (A)SDYM Equations, Integrability, and Lax Pair

In the previous talk in this Reading Group[1], we discussed the definition of non-abelian (A)SDYM instantons in four dimensions, i.e. finite-action non-singular solutions of the classical Yang-Mills equations of motion in Euclidean spacetime, and given a few examples. Let us study YM equations, their integrability, and (A)SD conditions on complexified Minkowski $\mathbb{M}_{\mathbb{C}}$.

2.1 YM field and (A)SD condition

Let us pick *halg-lightcone coordinates* Z^i and the volume form on $\mathbb{M}_{\mathbb{C}}$ given by

$$\mathcal{V} = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}. \tag{2.1}$$

The space of SD 2-forms is spanned by

$$\omega_1 = dw \wedge dz, \qquad \omega_2 = dw \wedge d\tilde{w} - dz \wedge d\tilde{z}, \qquad \omega_3 = d\tilde{z} \wedge d\tilde{w}. \tag{2.2}$$

[1] The previous talk presented Chapter 6 of [1] regarding Gauge Field Theory.

The space of ASD 2-forms is spanned by

$$\rho_1 = d\tilde{w} \wedge dz, \qquad \rho_2 = dw \wedge d\tilde{w} + dz \wedge d\tilde{z}, \qquad \rho_3 = d\tilde{z} \wedge dw. \tag{2.3}$$

Let $A = A_i dZ^i$ be a Lie-algebra $\mathfrak{g}$-valued gauge field, the covariant derivative be $D_i = \partial_i + A_i$, and the field strength $F_{ij} = [D_i, D_j]$. We can impose ASD and SD conditions on the 2-form F by requiring $F \wedge \omega_\ell = 0,\ \forall \ell = 1, 2, 3$ and $F \wedge \rho_\ell = 0,\ \forall \ell = 1, 2, 3$ respectively or

$$\text{ASD}: \quad F_{wz} = 0, \quad F_{w\tilde{w}} - F_{z\tilde{z}} = 0, \quad F_{\tilde{w}\tilde{z}} = 0 \tag{2.4}$$
$$\text{SD}: \quad F_{\tilde{w}z} = 0, \quad F_{w\tilde{w}} + F_{z\tilde{z}} = 0, \quad F_{w\tilde{z}} = 0. \tag{2.5}$$

We can see these conditions as yielding from a overdetermined linear system, as we will discuss in the next section.

2.2 Compatibility of Overdetermined Linear Systems as Integrability Condition

Let us first review in an example how Frobenius Theorem guarantees solutions to certain linear systems, ensuring integrability. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ and two 1-forms over the two-dimensional real space $A_x, A_y : \mathbb{R}^2_{(x,y)} \to \mathfrak{gl}(2, \mathbb{R}) \otimes \Lambda^1$, a two-dimensional vector $v : \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2$, and the overdetermined linear system:

$$D_x \equiv \partial_x v + A_x v = 0 \qquad D_y \equiv \partial_y v + A_y v = 0. \tag{2.6}$$

Let us review Frobenius Theorem in its version presented in [1] in Appendix C as Theorem C.2.5.

Theorem 2.1 (Frobenius Theorem)**.** *The necessary and sufficient condition for the existence of the unique solution $u^\alpha = u^\alpha(x)$ to*

$$\frac{\partial u^\beta}{\partial x^i} = \Psi_i^\beta(x, u) \tag{2.7}$$

for $i = 1, \dots, n$ and $\alpha, \beta = 1, \dots, N$ such that $u(x_0) = u_0$ for any initial data $(u_0, x_0) \in \mathbb{R}^{n+N}$ is that the relations

$$\frac{\partial \Psi_i^\alpha}{\partial x^j} - \frac{\partial \Psi_j^\beta}{\partial x^i} + \frac{\partial \Psi_i^\alpha}{\partial u^\beta} \psi_j^\beta - \frac{\partial \Psi_j^\alpha}{\partial u^\beta} \psi_i^\beta = 0 \tag{2.8}$$

hold.

Fixing $x^i = x$ and $x^j = y$, we get:

$$\left(\partial_x A_y - \partial_y A_x + [A_x, A_y]\right) v = 0. \tag{2.9}$$

Therefore, we obtain the zero-curvature condition:

$$F_{xy} = [D_x, D_y] = \partial_x A_y - \partial_y A_x + [A_x, A_y] = 0. \tag{2.10}$$

Let us know denote $g = (v_1\ v_2)$, where v_1, and v_2 are two independent solutions of the system in Eq. 2.6. Therefore, we can reconstruct the gauge field from g as

$$A_x = -(\partial_x g) g^{-1} \qquad A_y = -(\partial_y g) g^{-1} \qquad \Leftrightarrow \qquad A = -(dg) g^{-1} \tag{2.11}$$

which is a pure gauge field, as required by the zero-curvature condition.

This strategy to recover the gauge potential for a given problem by exploiting the compatibility condition prescribed by Frobenius Theorem can be replicated in the (A)SDYM equations case once we construct a pair of operators called the Lax Pair.

2.3 Lax Pair Formulation of ASDYM

Let $\lambda \in \mathbb{CP}^1$ be a spectral parameter. Let us introduce a pair of differential operators defined in *half-lightcone coordinates*:

$$L = D_{\tilde{z}} - \lambda D_w, \qquad M = D_{\tilde{w}} - \lambda D_z. \tag{2.12}$$

For some $\psi = \psi(z, \tilde{z}, w, \tilde{w}, \lambda) \in \mathfrak{g}$, they define an overdetermined linear system:

$$L\psi = 0 \ \wedge \ M\psi = 0. \tag{2.13}$$

The compatibility condition:

$$[L, M] = F_{\tilde{z}\tilde{w}} - \lambda(F_{w\tilde{w}} - F_{z\tilde{z}}) + \lambda^2 F_{wz} = 0 \tag{2.14}$$

is equivalent to the ASDYM equations, as it holds for any value of λ. From the equivalent formulation of the system

$$-(\partial_{\tilde{z}}\psi - \lambda\partial_w\psi)\psi^{-1} = A_{\tilde{z}} - \lambda A_w \ \wedge \ -(\partial_{\tilde{w}}\psi - \lambda\partial_z\psi)\psi^{-1} = A_{\tilde{w}} - \lambda A_z, \tag{2.15}$$

the gauge field A can be read off the RHS once ψ is known.

2.4 Lax Pair Formulation of SDYM

The same reasoning can be replicated for the SDYM case. Let $\hat{\lambda} \in \mathbb{CP}^1$ be a spectral parameter. Let us introduce a pair of differential operators defined in *half-lightcone coordinates*:

$$\hat{L} = D_z - \lambda D_w, \qquad \hat{M} = D_{\tilde{w}} - \lambda D_{\tilde{z}}. \tag{2.16}$$

For some $\psi = \psi(z, \tilde{z}, w, \tilde{w}, \lambda) \in \mathfrak{g}$, they define an overdetermined linear system:

$$\hat{L}\psi = 0 \ \wedge \ \hat{M}\psi = 0. \tag{2.17}$$

The compatibility condition:

$$[\hat{L}, \hat{M}] = F_{z\tilde{w}} - \lambda(F_{w\tilde{w}} + F_{z\tilde{z}}) + \lambda^2 F_{w\tilde{z}} = 0 \tag{2.18}$$

is equivalent to the SDYM equations, as it holds for any λ. From the equivalent formulation of the system

$$-(\partial_z\psi - \lambda\partial_w\psi)\psi^{-1} = A_z - \lambda A_w \ \wedge \ -(\partial_{\tilde{w}}\psi - \lambda\partial_{\tilde{z}}\psi)\psi^{-1} = A_{\tilde{w}} - \lambda A_{\tilde{z}}, \tag{2.19}$$

the gauge field A can be read off the RHS once ψ is known.

2.4.1 Geometric Interpretation of the (A)SDYM Lax Pair

The Lax operators for ASDYM and SDYM equations define a two-dimensional distribution depending on the spectral parameter $\lambda \in \mathbb{CP}^1$ and $\hat{\lambda} \in \mathbb{CP}^1$ respectively.

Geometrically, for fixed λ and $\hat{\lambda}$, the two pairs of vector fields

$$(\ell, m) : \ell = \partial_{\tilde{z}} - \lambda\partial_w, \quad m = \partial_{\tilde{w}} - \lambda\partial_z, \text{ and } \quad (\hat{\ell}, \hat{m}) : \hat{l} = \partial_z - \hat{\lambda}\partial_w, \quad \hat{m} = \partial_{\tilde{w}} - \hat{\lambda}\partial_{\tilde{z}} \tag{2.20}$$

span two two-planes in the tangent space of $\mathbb{M}_{\mathbb{C}}$. These planes are *totally null*, as

$$ASD: \quad \eta(\ell, \ell) = \eta(m, m) = \eta(\ell, m) = 0 \tag{2.21}$$

$$SD: \quad \eta(\hat{\ell}, \hat{\ell}) = \eta(\hat{m}, \hat{m}) = \eta(\hat{\ell}, \hat{m}) = 0. \tag{2.22}$$

We claim that the two-plane spanned by (ℓ, m) is SD, as $\mathcal{V}(\ell, m, \cdot, \cdot)$ is a SD 2-form on $\mathbb{M}_{\mathbb{C}}$, while conversely the two-plane spanned by $(\hat{\ell}, \hat{m})$ is ASD, as $\mathcal{V}(\hat{\ell}, \hat{m}, \cdot, \cdot)$ is an ASD 2-form on $\mathbb{M}_{\mathbb{C}}$. We also recover the operators by projecting the covariant derivative along the corresponding vector field:

$$L = \ell^i D_i, \quad M = m^i D_i, \hat{L} = \hat{\ell}^i D_i, \text{ and } \quad \hat{M} = \hat{m}^i D_i. \tag{2.23}$$

This discussion leads to two mirror statements.

Proposition 2.2 (ASDYM condition)**.** *The ASDYM condition $[L, M] = 0$ on a 1-form $A : \mathbb{M}_{\mathbb{C}} \to \mathfrak{g} \otimes \Lambda^1$ is equivalent to the vanishing of the YM curvature $F = dA + A \wedge A$ upon restriction to each SD totally null two-plane.*

Proposition 2.3 (SDYM condition)**.** *The SDYM condition $[\hat{L}, \hat{M}] = 0$ on a 1-form $A : \mathbb{M}_{\mathbb{C}} \to \mathfrak{g} \otimes \Lambda^1$ is equivalent to the vanishing of the YM curvature $F = dA + A \wedge A$ upon restriction to each ASD totally null two-plane.*

This observation is the underling property motivating the twistor approach to (A)SDYM equations.

3 The Twistor Correspondence

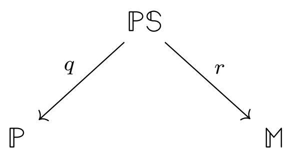
The aim of this section is introducing the Twistor Space through the Twistor Correspondence. In the following, some basics facts from Twistor Theory will be reviewed.

3.1 Double Fibration and Twistor Correspondence

First, let us define Twistor Space[2]. Let $\mathbb{T}$ be a four-complex-dimensions vector space equipped with an Hermitian form Φ of signature $(+, +, -, -)$. This space can be parametrised by $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$, where $\mu^{\dot{\alpha}}$ and λ_α are two spinors wit $SL(2, \mathbb{C})$ indices. To proceed in the definition we need to introduce the notion of flag manifold.

Definition 3.1 (Flag Manifold)**.** A flag manifold $\mathbb{F}_{d_1, \ldots d_r} = (L_1, \ldots, L_r)$ is a sequence of linear subspaces $L_1 \subset \cdots \subset L_r \subset \mathbb{T}$ with dimensions $\dim_{\mathbb{C}} L_j = d_j$.

Let us denote $\mathbb{PS} \equiv \mathbb{F}_{1,2} \cong G_{2,4}(\mathbb{C}) \times \mathbb{CP}^1$ the undotted projective spinor bundle, $\mathbb{P} \equiv \mathbb{F}_1 \cong \mathbb{CP}^3$, and $\mathbb{M} \equiv \mathbb{F}_2 \cong G_{2,4}(\mathbb{C})$ the complexified compactified conformally-flat spacetime, isomorphis to the 2-complex-dimensional linear subspaces inside $\mathbb{C}^4$. Let us interpret this construction: $\mathbb{PS}$ is the space of two-planes inside $\mathbb{C}^4$, and of the lines laying along them, $\mathbb{P}$ is the space of lines in $\mathbb{C}^4$, and $\mathbb{M}$ is the space of two-planes inside $\mathbb{C}^4$. The Twistor Correspondence is a double fibration



where the fibration $q\,(L_1, L_2) = L_1$ selects the line and $r\,(L_1, L_2) = L_2$ selects the plane.

On $\mathbb{PS} \cong G_{2,4}(\mathbb{C}) \times \mathbb{CP}^1$, we have coordinates $(x^{\alpha\dot{\alpha}}, \lambda_\alpha)$, where $x^{\alpha\dot{\alpha}}$ parametrises planes in $\mathbb{C}4$ and $\lambda \sim r\lambda$ for any $r \in \mathbb{C}^\times$ parametrises lines. The space $\mathbb{P} = \frac{\mathbb{PS}}{\delta_{\dot{\alpha}}} \cong \mathbb{CP}^3$ is defined by trivialising the Twistor Distribution $\delta_{\dot{\alpha}} = \lambda_\alpha \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$[3], imposing on $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$ projective rescaling $Z^A \sim rZ^A$ for any $r \in \mathbb{C}^\times$ and imposing the so-called incident relations $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$, which are the lineaer holomorphic

[2]This formulation is adapted from the one presented in [3].

[3]The Twistor Distribution spans totally-null ASD two-planes, called α-planes. Using the same coordinates as in the previous section, we can define $\delta_0 = \hat{\ell} = \partial_z - \hat{\lambda}\partial_w$, and $\delta_1 = \hat{m} = \partial_{\tilde{w}} - \hat{\lambda}\partial_{\tilde{z}}$. Alternatively, we could have also defined $\hat{\mathbb{P}} = \frac{\mathbb{PS}}{\hat{\delta}_\alpha}$ with $\hat{\delta}_\alpha = \mu^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$, imposing dual incident relations $\lambda^\alpha = \mu_{\dot{\alpha}} x^{\alpha\dot{\alpha}}$. In this case, the dual Twistor Distribution spans totally-null SD planes, called β-planes. Using the same coordinates as in the previous section, we can define $\hat{\delta}_0 = \ell = \partial_{\tilde{z}} - \lambda\partial_w$, and $\delta_1 = m = \partial_{\tilde{w}} - \lambda\partial_z$. This duality is exactly the same we observed in the previous section. We will come back on this point later on to see how it develops in Twistor Theory.

embedding of $\mathbb{CP}^1$, that is, Riemann spheres, in $\mathbb{P} \cong \mathbb{CP}^3$. Finally, on $\mathbb{M} \cong G_{2,4}(\mathbb{C})$, we have coordinates $x^{\alpha\dot{\alpha}}$.

The Twistor Correspondence consists in identifying:

- a point $x \in \mathbb{M} \longmapsto X \equiv q\left(r^{-1}(x)\right) \subset \mathbb{P}$ a line: $X \cong \mathbb{CP}1$,
- a point $Z \in \mathbb{P} \longmapsto \mathcal{Z} \equiv r\left(q^{-1}(X)\right) \subset \mathbb{M}$ an α-plane $\mathcal{Z} \cong \mathbb{CP}^2$.

To better understand the second instance of the correspondence, let us first consider the point $Z \in \mathbb{P}$ as intersection on two lines $X, Y \subset \mathbb{P}$, that is $Z = X \cap Y \mapsto \left(\mu^{\dot{\alpha}}, \lambda_\alpha\right)$; let us denot $x, y \in \mathbb{M}$ the points corresponding respectively to the lines $X, Y \subset \mathbb{P}$: the incident relations at each point read

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha, \qquad \mu^{\dot{\alpha}} = y^{\alpha\dot{\alpha}}\lambda_\alpha. \tag{3.1}$$

The existence of an intersection point requires $(x-y)^{\alpha\dot{\alpha}} = \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$, i.e. the points in spacetime to be null separated. The degrees of freedom $\tilde{\lambda}^{\dot{\alpha}}$ are the ones of the α-plane.

From this last observation, we learn that the Twistor Correspondence captures the conformal structure of $\mathbb{M}$: a point $x \in \mathbb{M}$ sits on the lightcone $L_x \subset \mathbb{M}$ of another point $x \in \mathbb{M}$ if and only if the corresponding lines $X, X' \subset \mathbb{P}$ intersect in $\mathbb{P}$.

In all truth, we still have not defined Twistor Space. In order to properly do so, one needs to discuss the conformal invariance of the structure we have built so fare: we will do this in the next section. For now, let us claim that Twistor Space is an open subset $\mathbb{PT} \underset{\text{open}}{\subset} \mathbb{P}$.

Before moving forward, let us state a fact and a definition. The fact: a holomorphic line embedded in $\mathbb{CP}^3$ can be specified by any two points that sit on it. In particular, we can represent any line $X \subset \mathbb{P}$ as a Klein Quadric $X^{[AB]} \in \mathcal{Q} = \left\{X \in \mathbb{CP}^5 \text{ such that } X^2 = 0\right\}$ specfied by two points $Z_1 = (x^{\alpha\dot{\alpha}}\lambda_{1\alpha}, \lambda_{1\alpha}), Z_2 = (x^{\alpha\dot{\alpha}}\lambda_{1\alpha}, \lambda_{2\alpha}) \in X \subset \mathbb{P}$ as the bi-twistor

$$X^{AB} = Z_1^{[A}Z_2^{B]} = \langle\lambda_1\lambda_2\rangle \begin{pmatrix} \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}x^2 & x^{\dot{\alpha}}_{\beta} \\ -x^{\dot{\beta}}_{\alpha} & \varepsilon_{\alpha\beta} \end{pmatrix}. \tag{3.2}$$

And, finally, the definition: we denote as $\mathbb{P}^\vee$ the dual space to $\mathbb{P}$. Coordinates on $\mathbb{P}^\vee$ are $W_A = \left(\tilde{\lambda}_{\dot{\alpha}}, \tilde{\mu}^\alpha\right)$. There exists a natural inner product

$$Z \cdot W = Z^A W_A = \left[\mu\tilde{\lambda}\right] + \langle\tilde{\mu}\lambda\rangle\,, \tag{3.3}$$

which is given by the Hermitian form Φ when we consider coordinates non-projectively as points in $\mathbb{C}^4$.

3.2 Conformal Structure and Twistor Space of $\mathbb{M}_\mathbb{C}$

The conformal group of complexified Minkowski space is $SL(4,\mathbb{C})$. There exists a representation of $SL(4,\mathbb{C})$ that acts linearly on $\mathbb{P}$: we can write the $SL(4,\mathbb{C})$-invariant $\varepsilon_{ABCD}Z_1^AZ_2^BZ_3^CZ_4^D$ for any $Z_1, Z_2, Z_3, Z_4 \in \mathbb{P}$ and as holomorphic generators $T^A_B = Z^A\frac{\partial}{\partial Z^B}$, which can be identified with:

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha\frac{\partial}{\partial\mu^{\dot{\alpha}}}\,, \qquad J_{\alpha\beta} = \lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)}}\,, \qquad \tilde{J}_{\dot{\alpha}\dot{\beta}} = \mu_{(\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\beta})}}\,, \tag{3.4}$$

$$K^{\alpha\dot{\alpha}} = \mu^{\dot{\alpha}}\frac{\partial}{\partial\lambda_\alpha}\,, \qquad D = \frac{1}{2}\left(\lambda_\alpha\frac{\partial}{\partial\lambda_\alpha} - \mu^{\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\alpha}}}\right)\,, \tag{3.5}$$

denoting $P_{\alpha\dot{\alpha}}$, $J_{\alpha\beta}$, $\tilde{J}_{\dot{\alpha}\dot{\beta}}$ as the generators of Lorentz boosts and rotations, $K^{\alpha\dot{\alpha}}$ the generator of special conformal transformations, and D the dilatation generator. It is easy to check that these operators obey the conformal algebra.

Any conformally flat metric can be brought to the form

$$ds^2 = \frac{dx^{\alpha\dot{\alpha}}dx_{\alpha\dot{\alpha}}}{(f(x))^2}, \tag{3.6}$$

where $f(x) = 1$ corresponds to $\mathbb{M}_{\mathbb{C}}$. Using the Twistor Correspondence and Eq. 3.2, we can rewrite the line element as:

$$ds^2 = \frac{\varepsilon_{ABCD}X^{AB}X^{CD}}{(I_{AB}X^{AB})^2} = \frac{\langle\lambda_1\lambda_2\rangle\, dx^{\alpha\dot{\alpha}}dx_{\alpha\dot{\alpha}}}{(I_{AB}X^{AB})^2} \tag{3.7}$$

for some projective-weight-zero fixed bitwistor I_{AB} called the *infinity twistor*. This is the ingredient that breaks conformal invariance, introducing a singularity in the metric at $I_{AB}X^{AB} = 0$, and allowing us to pick $\mathbb{M}_{\mathbb{C}}$ over all the conformally-flat spacetimes. Therefore, we can define Twistor Space as open subset $\mathbb{PT} \underset{\text{open}}{\subset} \mathbb{P}$ such that for any line $X \subset \mathbb{PT}$, $I_{AB}X^{AB} = 0$.

To go back to $\mathbb{M}_{\mathbb{C}}$, let

$$I_{AB} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{\alpha\beta} \end{pmatrix} \tag{3.8}$$

be the *infinity twistor*, which implies $I_{AB}X^{AB} = \langle\lambda_1\lambda_2\rangle$, and therefore $ds^2 = dx^{\alpha\dot{\alpha}}dx_{\alpha\dot{\alpha}}$ as desired.

Also, $I_{AB}X^{AB} = \langle\lambda_1\lambda_2\rangle = 0$ implies $\lambda_{1\alpha} = 0 = \lambda_{2\alpha}$. However, if we consider the incident relation $\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha$, we realise that for finite components of x one would get $Z^A = (0,0) \notin \mathbb{CP}^3$. Therefore, some components of x must be infinite: the *infinity twistor* really corresponds to some portion of the boundary of $\mathbb{M}_{\mathbb{C}}$. A more explicit definition of Twistor Space is therefore

$$\mathbb{PT} = \mathbb{P}\backslash\left\{Z^A \in \mathbb{P} \text{ such that } \lambda_\alpha \neq 0\right\}. \tag{3.9}$$

We also note that $I_{AB}I^{AB} = 0$, i.e. I is *simple*, therefore $I \subset \mathbb{P}\backslash\mathbb{PT}$ is a line, which corresponds to the point i_0, the spacelike infinity of $\mathbb{M}_{\mathbb{C}}$. Moreover, for any line $X \subset \mathbb{PT}$ such that $X \cap I \neq \varnothing$, the corresponding $x \in \mathbb{M}_{\mathbb{C}}$ is null separated from i_0, and therefore belongs to the null infinity $\mathscr{I}^{\pm}$.

3.3 Reality Structures

So far, we have only worked on complexified Minkowski spacetime $\mathbb{M}_{\mathbb{C}}$. To obtain real spacetimes with defined signature, we impose suitable reality conditions, i.e. we consider subspaces of $\mathbb{M}_{\mathbb{C}}$ which are real with respect to opportunely defined complex conjugation.

3.3.1 Lorentzian Signature

In order to recover $\mathbb{M} = \mathbb{R}^{1,3}$, we impose as reality condition the requirement for $x^{\alpha\dot{\alpha}}$ to be Hermitian:

$$x^{\alpha\dot{\alpha}} = \left(x^{\alpha\dot{\alpha}}\right)^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{x}^0 + \bar{x}^3 & \bar{x}^1 - i\bar{x}^2 \\ \bar{x}^1 + i\bar{x}^2 & \bar{x}^0 - \bar{x}^3 \end{pmatrix}, \tag{3.10}$$

where $\bar{x}$ is the usual complex conjugation of $\mathbb{C}$. This operation extends to spinors as

$$k^\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \overset{\dagger}{\longmapsto} \bar{k}^{\dot{\alpha}} = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}, \qquad \tilde{w}^{\dot{\alpha}} = \begin{pmatrix} c \\ d \end{pmatrix} \overset{\dagger}{\longmapsto} \tilde{\omega}^{\alpha} = \begin{pmatrix} \bar{c} \\ \bar{d} \end{pmatrix}. \tag{3.11}$$

We modify, for notation convenience, the incident relations to be

$$\text{on } \mathbb{PT},\ \mu^{\dot{\alpha}} = ix^{\alpha\dot{\alpha}}\lambda_\alpha, \qquad \text{and on } \mathbb{PT}^\vee,\ \tilde{\mu}^\alpha = -ix^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}. \tag{3.12}$$

As complex conjugation exchanges $SL(2,\mathbb{C})$ representations, we can assign to any point in $\mathbb{M}_{\mathbb{C}}$ twistor space $\mathbb{PT} \ni Z^A \overset{\dagger}{\longmapsto} \bar{Z}_A = \left(\bar{\lambda}_{\dot{\alpha}}, \bar{\mu}^\alpha\right) \in \mathbb{PT}^\vee$, a point in dual $\mathbb{M}_{\mathbb{C}}$ twistor space. The inner product $Z \cdot \bar{Z}$

is a $SU(2,2)$-invariant[4], as the spinors are Lorentzian-real Weyl spinor valued in $SU(2)$. How do we know what points of $\mathbb{M}_C$ correspond to $x^{\alpha\dot{\alpha}} \in \mathbb{M}$? Using incident relations, we obtain

$$Z \cdot \bar{Z} = i\left(x^{-}x^{\dagger}\right)^{\alpha\dot{\alpha}} \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}} = 0. \tag{3.13}$$

Therefore, we can define the twistor space of real Minkowski $\mathbb{M}$ as

$$\mathbb{PN} = \left\{ Z \in \mathbb{PT} \text{ such that } Z \cdot \bar{Z} = 0 \right\}, \tag{3.14}$$

which is also called space of null twistors. Any point $Z \in \mathbb{PT}$ corresponds to an α-plane in $\mathbb{M}_{\mathbb{C}}$, whose tangent vectors are parallel to λ_α. Moreover, any point in $Z \in \mathbb{PN}$ corresponds to a real unique geodesics $\lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$: this is the intersection between the α-plane associated to Z and the Lorentzian real slice $\mathbb{M}$. Lines in $\mathbb{PN}$ intersect only if the corresponding points in $\mathbb{M}$ are separated by a real null geodesic.

3.3.2 Euclidean Signature

In order to recover $\mathbb{M} = \mathbb{R}^4$, we impose as reality condition the requirement for $x^{\alpha\dot{\alpha}}$ to be real under complex conjugation given by:

$$x^{\alpha\dot{\alpha}} = \hat{x}^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{x}^0 - \bar{x}^3 & \bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{pmatrix}, \tag{3.15}$$

where $\bar{x}$ is the usual complex conjugation of $\mathbb{C}$. This operation extends to spinors as

$$k^\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \overset{\hat{}}{\longmapsto} \hat{k}^\alpha = \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}, \qquad \tilde{w}^{\dot{\alpha}} = \begin{pmatrix} c \\ d \end{pmatrix} \overset{\hat{}}{\longmapsto} \bar{\tilde{\omega}}^{\dot{\alpha}} = \begin{pmatrix} -\bar{d} \\ \bar{c} \end{pmatrix}. \tag{3.16}$$

The complex conjugation $\hat{}$ does not interchange $SL(2,\mathbb{C})$ representations, and since $\hat{\hat{}} = -1$, there no points in $\mathbb{PT}$ which are preserved by $\hat{}$. This statement is equivalent at claiming that there is no intersection between the α-plane associated to any $Z \in \mathbb{PT}$ and $\mathbb{R}^4$, that is there are no real null geodesics in Euclidean signature.

What about lines? We can structure lines of the kind $X^{AB} = Z^{[A}\hat{Z}^{B]}$, which are preserved by complex conjugation $\hat{}$. The possibility of defining such a bi-twistor means that:

- for any $Z \in \mathbb{PT}_E$ a point in the twistor space of $\mathbb{R}^4$, we can assign a line X^{AB} as above;
- by twistor correspondence, to any line $X \subset \mathbb{PT}_E$, we can assign a point $x \in \mathbb{R}^4$;
- any point $x \in \mathbb{R}^4$, by twistor correspondence, is related to $\mathbb{CP}^1$-worth points in $\mathbb{PT}_E$, that is the line X. To fix a point in $\mathbb{PT}_E$, one should first choose a point $x \in \mathbb{R}^4$ and then a λ^α on the $\mathbb{CP}^1$.

Therefore, despite conserving non-locality, the correspondence yields a $\mathbb{CP}^1$-bundle:

$$\mathbb{PT} \to \mathbb{R}^4, \qquad x^{\alpha\dot{\alpha}} = \frac{\hat{\mu}^{\dot{\alpha}}\lambda^\alpha - \mu^{\dot{\alpha}}\hat{\lambda}^\alpha}{\langle\lambda\hat{\lambda}\rangle}, \qquad \text{and} \quad \mathbb{PT}_E \cong \left(\mathbb{R}^4 \times \mathbb{CP}^1\right)_{(x^{\alpha\dot{\alpha}},\lambda_\alpha)} \cong \mathbb{PS}_E. \tag{3.17}$$

3.3.3 Split Signature

In order to recover $\mathbb{M} = \mathbb{R}^{2,2}$, we impose as reality condition the requirement for $x^{\alpha\dot{\alpha}}$ to be real under complex conjugation given by:

$$x^{\alpha\dot{\alpha}} = \overline{x^{\alpha\dot{\alpha}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{x}^0 + \bar{x}^3 & \bar{x}^1 + i\bar{x}^2 \\ \bar{x}^1 - i\bar{x}^2 & \bar{x}^0 - \bar{x}^3 \end{pmatrix}, \tag{3.18}$$

[4]Locally, $SU(2,2) \cong SO(4,2)$, which is the conformal group of $\mathbb{M}$.

where $\bar{x}$ is the usual complex conjugation of $\mathbb{C}$. This operation extends to spinors as

$$k^\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \overset{-}{\longmapsto} \overline{k^\alpha} = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}, \qquad \tilde{w}^{\dot{\alpha}} = \begin{pmatrix} c \\ d \end{pmatrix} \overset{-}{\longmapsto} \overline{\tilde{\omega}^{\dot{\alpha}}} = \begin{pmatrix} \bar{c} \\ \bar{d} \end{pmatrix}. \tag{3.19}$$

At the spinor level, this means that the symmetry group decomposes in $\mathrm{Spin}(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. In this case, projective twistor space admits a real slice isomorphic to $\mathbb{PT}_{\mathbb{R}} \subset \mathbb{RP}^3$: given a line $X \subset \mathbb{PT}_{\mathbb{R}}$, for any $Z \in X$, we have $Z = \bar{Z}$ when $x^{\alpha\dot{\alpha}} = \overline{x^{\alpha\dot{\alpha}}} \in \mathbb{R}^{2,2}$.

3.4 Complex Structures on Twistor Space

The study of (almost) complex structure is the core of complex geometry. Here, we only claim a few fundamental facts about complex structures on $\mathbb{PT} \underset{\text{open}}{\subset} \mathbb{CP}^3$. We can equip $\mathbb{PT} \underset{\text{open}}{\subset} \mathbb{CP}^3$ with an integrable complex structure, which yields:

- we can decompose any k-form along holomorphic and antiholomorphic coordinates as

$$\Omega^k(\mathbb{PT})_{\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}(\mathbb{PT}) \ni \omega = \omega_{a_1\dots a_p \bar{a}_1\dots\bar{a}_q} dz^1 \wedge \dots \wedge dz^p \wedge d\bar{z}^1 \wedge \dots \wedge d\bar{z}^q; \tag{3.20}$$

- we can define the so-called Dolbeault operator $\bar{\partial} : \Omega^{p,q}(\mathbb{PT}) \to \Omega^{p,q+1}(\mathbb{PT})$ such that $\bar{\partial}\big|_{\Omega^{p,q}(\mathbb{PT})} = \rho_{p,q+1} \circ d$, with projection $\rho_{p,q} : \Omega^k(\mathbb{PT}) \to \Omega^{p,q}(\mathbb{PT})$, and d standard exterior derivative;
- the Dolbeault operator is such that $\bar{\partial}^2 = 0$;
- locally, we can represent $\bar{\partial} = d\bar{Z}^A \frac{\partial}{\partial \bar{Z}^A}$, where what we mean by complex conjugation depends on the reality structure the we want to impose.

This gives rise to Dolbeault cohomology groups on twistor space $H^{p,q}_{\bar{\partial}}(\mathbb{PT})$.

We can construct an explicit bases for the antiholomorphic vector and $(0,1)-$forms on $\mathbb{PT}_E$:

$$T^{0,1}_{\mathbb{PT}_E} = \mathrm{span}\left\{ \bar{\partial}_0 = \langle \lambda\, \hat{\lambda} \rangle\, \lambda^\alpha \frac{\partial}{\partial \hat{\lambda}^\alpha},\ \bar{\partial}_{\dot{\alpha}} = \lambda^\alpha \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \right\} \tag{3.21}$$

$$\Omega^{0,1}(\mathbb{PT}_E) = \mathrm{span}\left\{ \bar{e}^0 = \frac{\langle \hat{\lambda}\, d\hat{\lambda} \rangle}{\langle \lambda\, \hat{\lambda} \rangle^2},\ \bar{e}^{\dot{\alpha}} = \frac{\hat{\lambda}_\alpha\, dx^{\alpha\dot{\alpha}}}{\langle \lambda\, \hat{\lambda} \rangle} \right\}. \tag{3.22}$$

The complex structure on $\mathbb{PT}_E$ is therefore given by $\bar{\partial} = \bar{e}^0\, \bar{\partial}_0 + \bar{e}^{\dot{\alpha}}\, \bar{\partial}_{\dot{\alpha}}$.

4 The Ward Correspondence

We now want to study the full non-linear (A)SD non-abelian YM equations. As we will see in this section, twistor methods have a lot to say in showing their integrability and finding ansatz to find YM instantons solutions. The main result we will show is the correspondence between SDYM instantons solutions for some gauge group $\mathcal{G}$ and an holomorphic vector bundle on $\mathbb{PT}$ with additional requirements needed to specify the chosen gauge group $\mathcal{G}$. How the ASD case con be recovered with the same but mirrored construction will be commented on at the of the section.

4.1 $GL(N,\mathbb{C})$-Gauge Theory on $\mathbb{M}_{\mathbb{C}}$ and on $\mathbb{PT}$

Let us consider a YM theory with non-abelian gauge group $\mathcal{G} = SL(N,\mathbb{C})$.

On Spacetime: Let us introduce the characters of this story on the spacetime $\mathbb{M}_{\mathbb{C}}$:

- Gauge Field: $A_a(x) : \mathbb{M}_{\mathbb{C}} \to \mathfrak{g} \otimes \Lambda^1 (\mathbb{M}_{\mathbb{C}})$ in the ajoint representation of the Lie-algebra $\mathfrak{g}$;
- Gauge Connection: $D_a = \partial_a + A_a$ that acts on objects $f(x)$ in the fundamental representation of $\mathfrak{g}$ as $D_a f(x) = \partial_a f(x) + A_a f(x)$ and on objects $\Psi(x)$ in the adjoint representation as $D_a \Psi = \partial_a \Psi + [A_a, \Psi]$;
- Curvature: $F_{ab} = [D_a, D_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b]$;
- Gauge Transformation Parameter $g : \mathbb{M}_{\mathbb{C}} \to \mathfrak{g}$ such that $A_a \xmapsto{g} g A_a g^{-1} - (\partial_a g)\, g^{-1}$ and $F_{ab} \xmapsto{g} g F_{ab} g^{-1}$.

On Twistor Space:

- Gauge Field: $a \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$;
- Covariant Almost-Complex Structure: $\bar{D} = \bar{\partial} + a$, a deformation of the Dolbeault operator via the Gauge Field, only along the antiholomorphic directions;
- Antiholomorphic Curvature: $F = \left[\bar{D}, \bar{D}\right] \in \Omega^{0,2}(\mathbb{PT}, \mathfrak{g})$;
- Gauge Transformation Parameter $\gamma \in \Omega^0 (\mathbb{PT}, \mathfrak{g})$ such that $\bar{D} \xmapsto{\gamma} \gamma \bar{D} \gamma^{-1}$ and $F \xmapsto{\gamma} \gamma F \gamma^{-1}$.

Finally, let us introduce some definitions.

Definition 4.1 (Holomorphic Vector Bundle over $\mathbb{PT}$ of rank N.)**.** A complex manifold E, an holomorphic projection $\pi : E \to \mathbb{PT}$ such that

- for any $Z \in \mathbb{PT}$, $\pi^{-1}(Z) \cong \mathbb{C}^N$;
- for any $Z \in \mathbb{PT}$, there exists a neighbourhood U_α and an homeomorphism χ_α such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \cong & U_\alpha \times \mathbb{C}^N \\ & {\scriptstyle \pi}\searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

- the patching matrix $\psi_{\alpha\beta} = \chi_\beta \circ \chi_\alpha^{-1} : U_\alpha \cap U_\beta \to GL(N, \mathbb{C})$ is holomorphic.

We claim two holomorphic vector bundles $(E, (\chi_\alpha, U_\alpha))$ and $\left(\tilde{E}, (\tilde{\chi}_\alpha, U_\alpha)\right)$ to be equivalent when $\tilde{\psi}_{\alpha\beta} = (H_\alpha)^{-1} \psi_{\alpha\beta} H_\alpha$ for some $H_\alpha = \chi_\alpha \circ \tilde{\chi}_\alpha^{-1}$ called *holomorphic frames*.

Definition 4.2 (Section)**.** A map $\Gamma : \mathbb{PT} \to E$ such that $\pi \circ \Gamma = \mathbb{1}_{\mathbb{PT}}$ is called a section of the vector bundle.

We consider a holomorphic vector bundle over $\mathbb{PT}$ of rank N, $E \to \mathbb{PT}$, such that

- locally, $E \cong \mathbb{C}^N \times \mathbb{PT}$, which means that the fiber at each point is given by $E|_Z \cong \mathbb{C}^N$;
- the restriction upon a line $X \cong \mathbb{CP}^1 \subset \mathbb{PT}$ is such that $E|_X \cong X \times \mathbb{C}^N$, i.e. trivial and the Chern class $c_1(E|_X) = 0$;
- holomorphicity of $(E, \bar{D})$: given $\bar{D} : \Gamma(E) \to \Omega^{0,1}(\mathbb{PT}, E)$ an almost complex structure on E, we require $\bar{D}^2 = 0$, i.e. $\bar{D}$ to be a integrable complex structure, which is equivalent to requiring $F = 0$.

Since $\mathrm{End}(E) \cong \mathfrak{gl}(N, \mathbb{C})$, the gauge group naturally encoded in E will be $GL(N, \mathbb{C})$.

4.1.1 From the Holomorphic Vector Bundles to the Gauge Fields

Let us work out explicitly one side of the correspondence in a constructive manner, with Euclidean reality conditions. Let $a = a_0 \bar{e}^0 + a_{\dot{\alpha}} \bar{e}^{\dot{\alpha}}$ be the $\mathfrak{gl}(N, \mathbb{C})$-gauge field on $\mathbb{PT}_E$, where $a_0, a_{\dot{\alpha}} : \mathbb{PT}_E \to \mathfrak{gl}(N, \mathbb{C})$ live in the adjoint representation and have homogeneous weight $+2$ and $+1$ respectively[5]. We have curvature

$$
\begin{aligned}
F &= \left(\bar{\partial}_0 a_{\dot{\alpha}} - \bar{\partial}_{\dot{\alpha}} a_0 - [a_{\dot{\alpha}},\, a_0]\right) \bar{e}^0 \wedge \bar{e}^{\dot{\alpha}} + \left(\bar{\partial}_{\dot{\alpha}} a_{\dot{\beta}} + \left[a_{\dot{\alpha}},\, a_{\dot{\beta}}\right]\right) \bar{e}^{\dot{\alpha}} \wedge \bar{e}^{\dot{\beta}} \\
&= \bar{D}\left(a_0 \bar{e}^0\right) + \left(\bar{\partial}_{\dot{\alpha}} a_{\dot{\beta}} + \left[a_{\dot{\alpha}},\, a_{\dot{\beta}}\right]\right) \bar{e}^{\dot{\alpha}} \wedge \bar{e}^{\dot{\beta}}.
\end{aligned} \tag{4.1}
$$

Let us check that the term $\bar{D}\left(a_0 \bar{e}^0\right)$ can be consistently removed by a gauge transformation. Being $a \in \Omega^{0,1}\left(\mathbb{PT}_E, \mathfrak{gl}(\mathbb{C}, N)\right)$, for a line $X \cong \mathbb{CP}^1 \subset \mathbb{PT}_E$, we have that $\bar{\partial}\big|_X a_0 = 0$; we can consider the gauge choice such that $\left(\bar{\partial}\big|_X\right)^{\text{adj}} a_0 = 0$ for any $X \cong \mathbb{CP}^1 \subset \mathbb{PT}_E$, which implies that a_0 is an harmonic function on $X \cong \mathbb{CP}^1$ and , by Hodge theorem, that $a_0 \in H^{0,1}_{\bar{\partial}}\left(X, \mathfrak{gl}(N, \mathbb{C})\right) = \varnothing$. Therefore, we can consistently set $a_0 = 0$.

The condition $F = 0$ yields

$$
\bar{\partial}_0 a_{\dot{\alpha}} = 0 \quad \wedge \quad \bar{\partial}_{\dot{\alpha}} a_{\dot{\beta}} + \left[a_{\dot{\alpha}},\, a_{\dot{\beta}}\right] = 0. \tag{4.2}
$$

Let $a(x, \lambda, \hat{\lambda})$ be holomorphic, Liouville theorem gives us $a_{\dot{\alpha}}(x, \lambda, \hat{\lambda}) = \lambda^\alpha A_{\alpha\dot{\alpha}}(x)$, with $A_{\alpha\dot{\alpha}}(x) : \mathbb{R}^4 \to \mathfrak{gl}(N, \mathbb{C}) \times \Lambda^{0,1}\left(\mathbb{R}^4\right)$ gauge field on the Euclidean space. Substituting this equation into 4.2, we find

$$
\varepsilon_{\dot{\alpha}\dot{\beta}} \lambda^\alpha \lambda^\beta F_{\alpha\beta} = 0 \quad \Leftrightarrow \quad F_{\alpha\beta} = 0, \tag{4.3}
$$

which means that $F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} + \varepsilon_{\alpha\beta} \tilde{F}_{\dot{\alpha}\dot{\beta}} = \varepsilon_{\alpha\beta} \tilde{F}_{\dot{\alpha}\dot{\beta}}$ is a SD $\mathfrak{gl}(N, \mathbb{C})$ curvature on $\mathbb{R}^4$. This kind of curvature satisfies both the Equation of Motion and the Bianchi identity of YM theory, i.e. it is a classical solution, and it is refer to as a $GL(N, \mathbb{C})$-SDYM instanton solution on $\mathbb{R}^4$.

4.1.2 From Gauge Fields to Holomorphic Vector Bundles

The converse is also true. Let us consider once again $\mathcal{G} = GL(N, \mathbb{C})$ and a SD gauge field on $\mathbb{M}_\mathbb{C}$: $F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta} \tilde{F}_{\alpha\dot{\alpha}\beta\dot{\beta}}$. For any $Z \in \mathbb{PT}$, we can assign a totally null α-plane in $\mathbb{M}_\mathbb{C}$[6]. Let v, w be two linearly independent vectors tangent to the α-plane, that is $v^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{v}^{\dot{\alpha}}$ and $w^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{w}^{\dot{\alpha}}$. The restriction upon the α-plane of the curvature is given by

$$
F_{\alpha\dot{\alpha}\beta\dot{\beta}}\Big|_{\alpha\text{-plane}} = v^{\gamma\dot{\gamma}} w^{\delta\dot{\delta}} F_{\gamma\dot{\gamma}\delta\dot{\delta}} \varepsilon_{\alpha\dot{\alpha}\beta\dot{\beta}} = \lambda^\gamma \lambda^\delta \tilde{v}^{\dot{\gamma}} \tilde{w}^{\dot{\beta}} \varepsilon_{\alpha\beta} \tilde{F}_{\dot{\gamma}\dot{\delta}} \varepsilon_{\alpha\dot{\alpha}\beta\dot{\beta}} = 0. \tag{4.4}
$$

We showed that F is flat upon restriction to α-planes[7]. Recall that points $Z \in \mathbb{PT}$ correspond to totally null α-planes in $M_\mathbb{C}$. If F vanishes when restricted to all such planes, the connection becomes flat along each α-plane. This allows us to claim that the space on covariantly constant function valued in the fundamental representation of $\mathfrak{gl}(N, \mathbb{C})$ is the same as the space of constant functions on α-planes, which means

$$
E|_Z = \left\{ \mathfrak{s}(x) : \mathbb{M}_\mathbb{C} \to \mathbb{C}^N \text{ such that } D_{\alpha\dot{\alpha}} \mathfrak{s}|_{\alpha\text{-plane}} = 0 \right\} \cong \mathbb{C}^N. \tag{4.5}
$$

Therefore, to any point $Z \in \mathbb{PT}$, which correspond to an α-plane in $\mathbb{M}_\mathbb{C}$, we can assign a vector bundle such that

- the bundle is topologically trivial upon restriction to lines, as defined point-by-point;

[5]The homogeneous weight are fixed so that a is weightless, given the dependence on λ_α of $\bar{e}^0$ and $\bar{e}^{\dot{\alpha}}$ in Eq. 3.22.

[6]One can think of this plane as the ASD planes spanned by $\hat{\ell}$ and $\hat{m}$, as defined in Eq. 2.20.

[7]One recognises here the condition of flatness upon restriction to totally null ASD two-planes, which is equivalent to SDYM condition as by Prop. 2.3.

- the bundle is holomorphic, as the whole construction is holomorphic.

We constructed the desired holomorphic vector bundle of rank N on $\mathbb{PT}$.

To conclude, a couple of remarks:

- This correspondence transforms a nonlinear system of PDEs into a problem in complex geometry.
- This construction can be extended to any $\mathcal{G}$ gauge group by adding further constraint on the vector bundle.
- Finally, one comment on the fact that we recovered SDYM instantons solutions. The choice of picking SD solutions was made in the definition of $\mathbb{P}$ as quotient space $\mathbb{P} = \frac{\mathbb{PS}}{\delta_{\dot{\alpha}}} \cong \mathbb{CP}^3$, by trivialising the Twistor Distribution $\delta_{\dot{\alpha}} = \lambda_\alpha \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$, spanned by $\delta_0 = \hat{\ell} = \partial_z - \lambda\partial_w$ and $\delta_1 = \hat{m} = \partial_{\tilde{w}} - \lambda\partial_{\tilde{z}}$. As already mentioned in a previous footnote, alternatively, we could have also defined $\hat{\mathbb{P}} = \frac{\mathbb{PS}}{\hat{\delta}_\alpha}$ with $\hat{\delta}_\alpha = \mu^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$, imposing alternative incident relations $\lambda^\alpha = \mu_{\dot{\alpha}} x^{\alpha\dot{\alpha}}$. In this case, the dual Twistor Distribution spans totally-null SD planes, called β-planes, generated by ℓ and m, where $\hat{\delta}_0 = \ell = \partial_{\tilde{z}} - \lambda\partial_w$, $\delta_1 = m = \partial_{\tilde{w}} - \lambda\partial_z$. The same construction that we worked out in the last sections can be repeated in the alternative case, showing a correspondence between ASDYM instantons solutions and holomorphic vector bundles over the alternatively-defined twistor space $\hat{\mathbb{PT}}$.

5 Strategy for Reconstruction of the Gauge Field

In this final section, we work out a strategy for the reconstruction of the $GL(N,\mathbb{C})$-gauge field. The main tool that simplifies the problem here is a result from complex analysis.

Theorem 5.1 (Birkhoff–Grothendieck)**.** *A rank N holomorphic vector bundle $E \to \mathbb{CP}^1$ is isomorphic to the direct sum of line bundles*[8]

$$\bigoplus_{i=1}^{N} \mathcal{O}(m_i), \tag{5.1}$$

for some integers m_i.

Let U_0 and U_1 be a open covering of any twistor line $X \cong \mathbb{CP}^1$ such that $\lambda_0 \neq 0$ and $\lambda_1 \neq 0$ on the respective open set. The transition map between the coordinates is defined on $U_0 \cap U_1 = \mathbb{C}^\times$ as $\lambda = \frac{\lambda_0}{\lambda_1}$. The patching matrix can therefore be written through *holomorphic frames* $\tilde{H} : U_0 \to GL(N,\mathbb{C})$ and $H : U_1 \to GL(N,\mathbb{C})$ as $\psi = \tilde{H}\,\mathrm{diag}\,(\lambda^{-m_1},\ldots,\lambda^{-m_N})\,H^{-1}$.

In the context of the Ward correspondence, we require triviality along twistor lines, which implies that the patching matrix is such that $\psi = \tilde{H}\mathbb{1}H = \tilde{H}\mathbb{1}H$, that is

$$E|_X \cong \mathcal{O}^{\oplus N}, \qquad \mathcal{O} = \mathcal{O}(0). \tag{5.2}$$

The patching matrix ψ is well-defined on lines inside $\mathbb{PT}$. Therefore, $\delta_{\dot{\alpha}}\psi = 0$:

$$\lambda^\alpha \partial_{\alpha\dot{\alpha}}\psi = \lambda^\alpha \left[\left(\partial_{\alpha\dot{\alpha}}\tilde{H} \right) H^{-1} + \tilde{H}\partial_{\alpha\dot{\alpha}}H^{-1} \right] = 0 \quad \Leftrightarrow \quad \lambda^\alpha \left(\partial_{\alpha\dot{\alpha}}\tilde{H} \right) H^{-1} = -\lambda^\alpha \tilde{H}\partial_{\alpha\dot{\alpha}}H^{-1}. \tag{5.3}$$

This gives an object which is of homogeneous weight $+1$ and holomorphic; by Liouville theorem we deduce

$$\lambda^\alpha \tilde{H}^{-1}\partial_{\alpha\dot{\alpha}}\tilde{H} = -\lambda^\alpha \left(\partial_{\alpha\dot{\alpha}}H^{-1} \right) H = \lambda^\alpha H^{-1}\partial_{\alpha\dot{\alpha}}H = \lambda^\alpha A_{\alpha\dot{\alpha}}(x). \tag{5.4}$$

By contracting with $\lambda^\beta \partial^{\dot{\beta}}_\beta$, we find

$$\partial^{\dot{\alpha}}_{(\beta} A_{\alpha)\dot{\alpha}} + A^{\dot{\alpha}}_{(\beta} A_{\alpha)\dot{\alpha}} = 0 \quad \Leftrightarrow \quad \left[D_{\dot{\alpha}(\alpha}, D_{\beta)\dot{\beta}} \right] = 0 \quad \Leftrightarrow \quad \varepsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} = 0, \tag{5.5}$$

[8] A line bundle is simply a vector bundle such that the rank is $N = 1$.

which is the SDYM condition. We have shown that H and $\tilde{H}$ are in the kernel of the Lax pair as defined in Equation , that is:

$$\hat{L}_{\dot{\alpha}} H = \lambda^{\alpha} D_{\alpha\dot{\alpha}} H = \lambda^{\alpha}(\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}})H = \delta_{\dot{\alpha}} H + H^{-1}\left(\delta_{\dot{\alpha}} H\right) H = 0; \tag{5.6}$$

$$\hat{L}_{\dot{\alpha}} \tilde{H} = \lambda^{\alpha} D_{\alpha\dot{\alpha}} \tilde{H} = \lambda^{\alpha}(\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}})\tilde{H} = \delta_{\dot{\alpha}} \tilde{H} + \tilde{H}^{-1}\left(\delta_{\dot{\alpha}} \tilde{H}\right) \tilde{H} = 0. \tag{5.7}$$

In practice, to recover the SDYM gauge field $A_{\alpha\dot{\alpha}}$ from the *holomorphic frames* H and $\tilde{H}$ we can evaluate Eq. 5.4 for $\lambda^{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\alpha}$ and $\lambda^{\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\alpha}$.

Once again, we could have carried out the mirrored procedure considering lines in $\hat{\mathbb{PT}}$. Therefore, $\hat{\delta}_{\alpha}\psi = 0$:

$$\mu^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi = \mu^{\dot{\alpha}} \left[\left(\partial_{\alpha\dot{\alpha}} \tilde{H}\right) H^{-1} + \tilde{H} \partial_{\alpha\dot{\alpha}} H^{-1}\right] = 0 \quad \Leftrightarrow \quad \mu^{\dot{\alpha}} \left(\partial_{\alpha\dot{\alpha}} \tilde{H}\right) H^{-1} = -\mu^{\dot{\alpha}} \tilde{H} \partial_{\alpha\dot{\alpha}} H^{-1}. \tag{5.8}$$

This gives an object which is of homogeneous weight +1 and holomorphic; by Liouville theorem we deduce

$$\mu^{\dot{\alpha}} \tilde{H}^{-1} \partial_{\alpha\dot{\alpha}} \tilde{H} = -\mu^{\dot{\alpha}} \left(\partial_{\alpha\dot{\alpha}} H^{-1}\right) H = \mu^{\dot{\alpha}} H^{-1} \partial_{\alpha\dot{\alpha}} H = \mu^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x). \tag{5.9}$$

By contracting with $\mu^{\dot{\beta}} \partial^{\beta}_{\dot{\beta}}$, we find

$$\partial^{\alpha}_{(\dot{\beta}} A_{\dot{\alpha})\alpha} + A^{\alpha}_{(\dot{\beta}} A_{\dot{\alpha})\alpha} = 0 \quad \Leftrightarrow \quad \left[D_{\alpha(\dot{\alpha}}, D_{\dot{\beta})\beta}\right] = 0 \quad \Leftrightarrow \quad \varepsilon_{\alpha\beta} \tilde{F}_{\dot{\alpha}\dot{\beta}} = 0, \tag{5.10}$$

which is the ASDYM condition. We have shown that H and $\tilde{H}$ are in the kernel of the Lax pair as defined in Equation , that is:

$$L_{\dot{\alpha}} H = \mu^{\dot{\alpha}} D_{\alpha\dot{\alpha}} H = \mu^{\dot{\alpha}}(\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}})H = \delta_{\alpha} H + H^{-1}\left(\delta_{\alpha} H\right) H = 0; \tag{5.11}$$

$$L_{\dot{\alpha}} \tilde{H} = \mu^{\dot{\alpha}} D_{\alpha\dot{\alpha}} \tilde{H} = \mu^{\dot{\alpha}}(\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}})\tilde{H} = \delta_{\alpha} \tilde{H} + \tilde{H}^{-1}\left(\delta_{\alpha} \tilde{H}\right) \tilde{H} = 0. \tag{5.12}$$

To conclude, in practice, to recover the ASDYM gauge field $A_{\alpha\dot{\alpha}}$ from the *holomorphic frames* H and $\tilde{H}$ we can evaluate Eq. 5.9 for $\mu^{\dot{\alpha}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\dot{\alpha}}$ and $\mu^{\dot{\alpha}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\dot{\alpha}}$.

5.1 An Example: ASD $GL(1, \mathbb{C})$ solutions

If we consider $\mathcal{G} = GL(1, \mathbb{C})$, we have that E is a line bundle such that $c_1(E) = 0$, as $E|_X = \mathcal{O}$ for any twistor line. The patching matrix is nowhere vanishing, therefore $\psi = e^f$ and we can also parametrise the nowhere vanishing holomorphic frames $H = e^h$, $\tilde{H} = e^{\tilde{h}}$, with $f = \tilde{h} - h$, where $f, h, \tilde{h}$ are holomorphic functions of homogeneous degree 0.

Consider the gauge degree of freedom $[i] \in \mathbb{CP}^1$, and some $F = f^{\bar{\alpha}}(x \cdot \lambda, \lambda) d\bar{\lambda}_{\bar{\alpha}} \in H^{0,1}_{\bar{\partial}}(\mathbb{PT}, \mathcal{O})$ and for any point on $\mathbb{CP}^1$ we can choose the rescaling appropriately so that only one component on $f^{\bar{\alpha}} d\bar{\lambda}_{\bar{\alpha}}$ is nonvanishing and fix it to be f. Since $F|_{X \cong \mathbb{CP}^1} \in H^{0,1}_{\bar{\partial}}(X, \mathcal{O}) = \varnothing$, there is a unique splitting

$$h = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\langle \lambda i \rangle}{\langle \lambda \rho \rangle \langle i \rho \rangle} f\left(x_{\alpha\dot{\alpha}} \rho^{\alpha}, \rho^{\alpha}\right) \langle \rho d\rho \rangle, \qquad \tilde{h} = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}} \frac{\langle \lambda i \rangle}{\langle \lambda \rho \rangle \langle i \rho \rangle} f\left(x_{\alpha\dot{\alpha}} \rho^{\alpha}, \rho^{\alpha}\right) \langle \rho d\rho \rangle, \tag{5.13}$$

where $\Gamma, \tilde{\Gamma} \subset \mathbb{C}$ are such that $\Gamma - \tilde{\Gamma}$ is enclosing $\rho = \lambda$. Imposing incident relation $\partial_{\alpha\dot{\alpha}} = \rho_{\alpha} \partial_{\dot{\alpha}}$ in Eq. 5.4, one finds that

$$A_{\alpha\dot{\alpha}} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{i_{\alpha}}{\langle \rho i \rangle} \frac{\partial f(x_{\alpha\dot{\alpha}} \rho^{\alpha}, \rho^{\alpha})}{\partial \mu^{\dot{\alpha}}} \langle \rho d\rho \rangle. \tag{5.14}$$

Since $GL(1,\mathbb{C})$ is Abelian, the equations $A_{\alpha\dot{\alpha}}$ obeys are linear. We are after the ASDYM solutions; therefore, Eq. 5.14 is nothing but the instance of the Penrose Transform for $h = 1$, that is an isomorphism between the set of all helicity $h \in \mathbb{Z}/2$ solutions of the linearised zero-rest-mass fields equations on $\mathbb{M}_{\mathbb{C}}$ and $H^{0,1}_{\bar{\partial}}(\mathbb{PT}, \mathcal{O}(2h-2))$. More information on the Penrose Transform can be found in [2].

In the non-Abelian case, the Ward Correspondence constructions is sensitive to non linear part of the equation of motions and Bianchi identity, while the Penrose Transform is not. In *almost all* the cases, an *holomorphic frame* $H(x,\lambda)$ such that, for any twistor line X, it holomorphically trivialises the deformation to the complex structure given by the a gauge field, i.e. $H^{-1}(x,\lambda)\circ\left(\bar{\partial}+a\right)\big|_X\circ H(x,\lambda) = \bar{\partial}\big|_X$, can be found. The hard part of finding YM instantons solutions via Ward Correspondence is exactly identifying the correct *holomorphic frame*. Some ansatz have been found, e.g. see [4] by Ward.

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