The Inversion Formula ~ Starting the Conformal THERMAL Bootstrap

didn't get here :(

• Motivation

→ Bootstrap @ T=0 : Aim to constrain CFT data $\{\{\Delta_\sigma, J_\sigma\}, f_{\sigma_1\sigma_2\sigma_3}\}$

Conformal dimension

structure constants (coefficients in OPE)

spin

↳ How?

- Conformal Symmetry (1)
- OPE (2)
- Crossing Symmetry (3)

→ Bootstrap @ $T = 1/\beta$ (i.e finite & non-zero)

↳ How?

- Quantize on $\mathcal{M}^d_\beta = S^1 \times \mathbb{R}^{d-1}$ where S^1 corresponds to the Euclidean time

↳ Why?

- Quantum Critical Points always have non-zero temperature in the laboratory.
- For $d > 2$, $\mathcal{M}^d_\beta \not\cong \mathbb{R}^d$ (Globally) ~ Ideally, any non-pert. solution of a QFT should describe its observables on arbitrary manifolds.
- In holography context, FTCFT's are dual to AdS black holes.

Background/Overveiw

→ From last week: $\mathbb{R}^{d-1} \times S^1$ breaks conformal symmetry (1) but we have spatial rotations, translations, 'scale invariance'.

→ $\mathcal{M}^d_\beta$ is conformally flat, locally $\mathcal{M}^d_\beta \simeq \mathbb{R}^d$, so we can compute two-point functions using the OPE, assuming Δx is sufficiently small.

↳ But we have new data, thermal one-point functions.

↳ 1-point fnct.

$$\langle \mathcal{O} \rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}}$$

scalar operator

- Here the β dependence is fixed by scale symmetry
- The coeff. $b_\mathcal{O}$ is not fixed

↳ 2-point fnct.

$$g(\tau) \equiv \langle \phi(\tau)\, \phi(0) \rangle_\beta \sim \frac{1}{|\tau|^{2\Delta_\phi}} \sum_{\mathcal{O} \in \phi \times \phi} \frac{f_{\phi\phi\mathcal{O}}\, b_\mathcal{O}}{c_\mathcal{O}} \left| \frac{\tau}{\beta} \right|^{\Delta_\mathcal{O}}$$

OPE coeff. of $\mathcal{O}$

Scaling dim of $\mathcal{O}$

2-point coefficient in vacuum

! Only chosen separation in in the circle direction of distance τ

→ We have the KMS condition: $g(\tau) \equiv g(\beta - \tau)$

Q: Does $g(\tau)$ given satisfy the KMS condition?

A: No!

→ From KMS we retrieve a "thermal crossing equation" → (3)

↳ This constrains $b_\mathcal{O}$ in terms of CFT data, → $f_{\phi\phi\mathcal{O}}$ and $\Delta_\mathcal{O}$

↳ From the limit $S^1 \times S^{d-1} \to S^1 \times \mathbb{R}^{d-1}$ we can see the "crossing symmetry" ~ usual one of crossing symmetry for 4-point functions where we sum over some external operators

↳ so $\sum_{12} \sum_{34} \langle O_1 O_2 O_3 O_4 \rangle$
$= \langle O_1 O_2 \rangle \langle O_3 O_4 \rangle$?

→ Goal: The thermal bootstrap ⟷ Compute all b_O using KMS and zero temp CFT data

Ch1 : Zero-Temp Recap

- Any CFT T_n on $\mathbb{R}^d$ can be computed using OPE.
- E.g.:

$$\langle O_1(x_1) O_2(x_2) \cdots O_n(x_n) \rangle = \sum_{k_1} C_{12k_1} \langle O_{k_1} O_3 \cdots O_n \rangle$$

$$= \sum_{k_1} \cdots \sum_{k_{n-1}} C_{12k_1} C_{k_1 3 k_2} \cdots C_{k_{n-2} n k_{n-1}} \langle O_{k_{n-1}} \rangle$$

- Consider 3-point:

⤳ OPE defines an algebra $O_i \times O_j = \sum_k C_{ijk} O_k$

⤳ Associative: $(O_i O_j) O_k = O_i (O_j O_k)$

$$\langle O_1 O_2 O_3 \rangle = \sum_{k_1} \frac{C_{12k_1}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_{k_1}}} \langle O_{k_1} O_3 \rangle$$

→ $O_{k_1} = O_3 = O$

$$= \sum_{k_1} \frac{C_{12k_1}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_{k_1}}} \frac{\delta_{k_1 3}}{|x_{23}|^{2\Delta_3}}$$

$$= \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{2\Delta_3}} \quad \circledast$$

- Conformal symmetry fixes 3-point:

$$\langle O_1 O_2 O_2 \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{31}|^{\Delta_1+\Delta_3-\Delta_2}}$$

→ In limit $x_1 \to x_2$ do we recover $\circledast$?

$$\simeq \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{2\Delta_3}}$$

$|x_1 - x_2| \ll |x_1 - x_3|, |x_2 - x_3|$

$\Rightarrow |x_1 - x_3| = |\underbrace{(x_1 - x_2)}_{\epsilon \ll} + \underbrace{(x_2 - x_3)}| \simeq |x_2 - x_3|$

So $|x_{23}| \simeq |x_{31}| \to |x_{13}|^{\Delta_1+\Delta_3-\Delta_2} \times |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} \simeq |x_{23}|^{2\Delta_3}$ ✓ as expected :)

- First interesting case @ 4pt:

$$\langle O_1 O_2 O_3 O_4 \rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2} |x_{34}|^{\Delta_3+\Delta_4}} G(u,v)$$

→ contains all dynamical info

where u, v are cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

↳ Now consider OPE

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \sim \sum_{\mathcal{O}} C_{12\mathcal{O}} \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}} \underbrace{\langle \mathcal{O}(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle}_{\text{fixed by conformal symmetry}}$$

$$\sim \sum_{\mathcal{O}} C_{12\mathcal{O}} \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}} \frac{C_{\mathcal{O}34}}{|x_{23}|^{\Delta_{\mathcal{O}}+\Delta_3-\Delta_4}|x_{24}|^{\Delta_{\mathcal{O}}+\Delta_4-\Delta_3}|x_{34}|^{\Delta_3+\Delta_4-\Delta_{\mathcal{O}}}}$$

- So $G(u,v) = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{\mathcal{O}34} G_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v;\Delta_i)$

CROSSING EQUATION

$$\sum_{\mathcal{O}} C_{12\mathcal{O}} C_{\mathcal{O}34} G_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(u,v) = \sum_{\mathcal{O}} C_{13\mathcal{O}} C_{\mathcal{O}24} G_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(v,u;\Delta_i) = \sum_{\mathcal{O}} C_{14\mathcal{O}} C_{\mathcal{O}23} G_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}\left(\tfrac{u}{v},\tfrac{1}{v}\right)$$

Ch. 2 Non-Zero Temp

- But @ $T = 1/\beta$ we know $\langle \mathcal{O}\rangle \neq \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } \mathcal{O} = 1 \end{cases}$ it's more complicated

▷ Assume unitarity so only the unit op. has scaling dim = 0

1) Non-Unit operators can have non-zero one-point functions.
2) Depending on the configuration of operator insertions, it may not always be possible to preform the OPE.
   ↳ Specifically, to compute $\mathcal{O}_i \times \mathcal{O}_j$ we must find a sphere containing only $\mathcal{O}_i$ and $\mathcal{O}_j$ whose interior is flat (possibly after a Weyl transform... so conformally flat?), but this may not be possible for a $\mathcal{M}^d$

- Here $\mathcal{M}^d = S^1_\beta \times \mathbb{R}^{d-1}$, co-ords $x = (\tau, \mathbf{x})$ where τ is periodic $\tau \sim \tau + \beta$

- So what symmetries constrain 1-point fncts?

1. Translation invariance
   - Implies descendents have vanishing 1-pt. fncts.

$$\langle P^N \mathcal{O}(x)\rangle_\beta = \partial^N \langle \mathcal{O}(x)\rangle_\beta = \partial^N \langle \mathcal{O}(0)\rangle_\beta = 0$$

E.g $\langle \mathcal{O}^a(x)\mathcal{O}^b(0)\rangle = \frac{I^{ab}(x)}{|x|^{2\Delta}}$ where if $\mathcal{O}$ = scalar then $I^{ab} = \delta^{ab} = 1$

Conformal group $SO(d+1,1)$, $\mathcal{O}(x)$ transforms simply under conformal transformations + annihillated by K_μ (gen. SC'g'op)

- Under rotations ($SO(d)$) the comp. of $\mathcal{O}(x)$ can transform in a rep. ρ of $SO(d)$. ρ = scalar, vector, STT's, ...

2. Focus on symmetry/primary operators: $\mathcal{O}$ w/ dimension Δ and $SO(d)$ rep. ρ
   - For the geometry $S^1 \times \mathbb{R}^{d-1}$ we have $SO(d-1)$
   - As well as discrete symmetry where $\tau \longleftrightarrow -\tau$ → Note: Not guaranteed parity ∴ accompany discrete time-reversal w/ reflection in <u>one</u> direction on $\mathbb{R}^{d-1}$

⟹ These symmetries together give symmetry group $O(d-1) \subset SO(d)$

↑ If we have parity then $O(d-1) \times \mathbb{Z}_2$

• So for $\langle \mathcal{O} \rangle_\beta \neq 0$ we need the restriction of ρ under $O(d-1) \subset SO(d)$ to contain trivial rep.

$$\mathrm{Res}^{SO(d)}_{O(d-1)}\, \rho \supset \mathbf{1}$$

restriction

trivial rep.

• Rep. ρ of group G $\quad \rho : G \longrightarrow GL(V)$ where V is a vector space s.t
$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \quad g_1, g_2 \in G$

• Suppose $H \subset G$ is a subgroup, to restrict ρ to H means
$\rho|_H : H \longrightarrow GL(V)$, $\rho|_H$

• **Therefore**:

Trivial rep requirements mean ρ = STT, with <u>even</u> spin J. $\therefore \langle \mathcal{O}^{\mu_1 \dots \mu_J} \rangle \sim (e^{\mu_1} \dots e^{\mu_J} - \text{traces})$

unit vector in τ direction

• Anti-symmetric tensor: $\mathcal{O}_{[\mu,\nu]}$
  • restricts to $SO(d-1)$ (transverse rot.)
  • doesn't contain triv. rep. since for every comp. $\exists$ transverse index

• Spinors
  - No
  - No

NOTE: In thesis they use broken ward identities from dilatons and boosts to provide these constraints... I think this is arguing the same point?

• Same result $\rightarrow$ plus interesting constraint in scalar OPE's $\rightarrow$ No vectors, no AS tensors ...

$$\langle \mathcal{O}(x) \rangle_\beta = \frac{\beta}{\Delta_\mathcal{O}} \int d^{d-1}y \, \langle T^{00}(0,y)\, \mathcal{O}(x) \rangle_\beta \quad \text{only non-trivial}$$

↳ can provide non-triv. const. on $C_\mathcal{O}$ of $T=0$ theory

3. Dimensional analysis

• Primary operator $[\mathcal{O}] = \text{length}^{-\Delta}$

• The only length scale is given by β $\therefore$ $\langle \mathcal{O} \rangle \sim \dfrac{1}{\beta^\Delta}$

from before

$\rightarrow$ All together:

$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J} \rangle_\beta \sim \frac{b_\mathcal{O}}{\beta^\Delta} \left(e^{\mu_1} \dots e^{\mu_J} - \text{traces} \right)$$

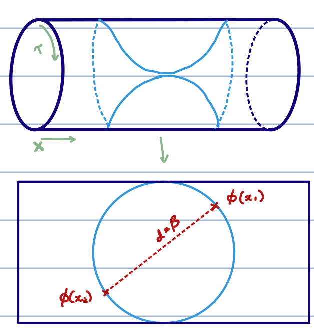
Ch 3. Encoding '$b_\mathcal{O}$' in OPE

~ NOTE: Similarly 2-pt. functions of non-identical operators might not vanish $\leadsto$ Interesting?

• Focus on identical op's atm.

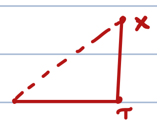
S3.1: When are OPE's valid?

$\boxed{\bullet \; g(\tau, \mathbf{x}) = \langle \phi(x)\, \phi(0) \rangle_\beta}$ the OPE is valid when both operators lie within a sphere whose interior is flat.



Largest sphere when $d < \beta$ so we don't get <u>winding</u> (curvature) or self intersection.

• Largest distance, $x_1 = (0,0)$, $x_2 = (\tau, \mathbf{X})$ gives $|x_2| = \sqrt{\tau^2 + \mathbf{X}^2} < \beta$



→ $\boxed{\text{Radius of convergence: } |x| = \sqrt{\tau^2 + \mathbf{x}^2} < \beta}$

Compute 3-point $\langle\phi\phi\mathcal{O}\rangle \sim C_{\phi\phi\mathcal{O}}$ $\langle\mathcal{O}\mathcal{O}\rangle \sim C_{\phi\phi\mathcal{O}}\, C_{\mathcal{O}}$

A: But $f_{\phi\phi\mathcal{O}} = C_{\phi\phi\mathcal{O}}\, C_{\mathcal{O}}$

Q: Is $C_{\phi\phi\mathcal{O}} = \dfrac{f_{\phi\phi\mathcal{O}}}{C_{\mathcal{O}}}$? where $C_{\mathcal{O}}$ is coeff of 2pt. zero temp coefficients

S3.2 Assume radius of convergence

- Use the OPE to get:

$$g(\tau,\mathbf{x}) = \langle\phi(\tau,\mathbf{x})\,\phi(0)\rangle_\beta = \sum_{\mathcal{O}\in\phi\times\phi} \frac{C_{\phi\phi\mathcal{O}}}{|x|^{2\Delta_\phi-\Delta_{\mathcal{O}}+J}}\, x_{\mu_1}\cdots x_{\mu_J}\, \langle\mathcal{O}^{\mu_1\cdots\mu_J}(0)\rangle_\beta + \text{descendants}$$

which we know go to zero!

$$= \sum_{\mathcal{O}\in\phi\times\phi} \frac{f_{\phi\phi\mathcal{O}}}{C_{\mathcal{O}}}\, |x|^{\Delta_{\mathcal{O}}-2\Delta_\phi-J}\, x_{\mu_1}\cdots x_{\mu_J}\, \frac{b_{\mathcal{O}}}{\beta^\Delta}\left(e^{\mu_1}\cdots e^{\mu_J} - \text{traces}\right)$$

- Index contraction: "Gegenbauer Polynomial"

$$|x|^{-J}(x_{\mu_1}\cdots x_{\mu_J})(e^{\mu_1}\cdots e^{\mu_J} - \text{traces}) = \frac{J!}{2^J(\nu)_J}\, C_J^{(\nu)}(\eta)$$

where $\nu = \dfrac{d-2}{2}$, $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ ~ Pochhammer symbol , $\eta = \dfrac{\tau}{|x|}$

Therefore

$$g(\tau,\mathbf{x}) = \sum_{\mathcal{O}\in\phi\times\phi} \frac{a_{\mathcal{O}}}{\beta^\Delta}\, C_J^{(\nu)}(\eta)\, |x|^{\Delta-2\Delta_\phi}$$

$$\rightarrow a_{\mathcal{O}} \equiv \frac{f_{\phi\phi\mathcal{O}}}{C_{\mathcal{O}}}\, \frac{J!}{2^J(\nu)_J}$$

- Consider $C_J^{(\nu)}(\eta)\,|x|^{\Delta-2\Delta_\phi}$ to be a 2. pt. conformal block on $S^1\times\mathbb{R}^{d-1}$

Cht. 4 Some Examples

Sc. 4.1 Free Energy

- Important thermal one-point coefficients is b_T, associated w/ $T^{\mu\nu} = \mathcal{O}$, which can be related to the free energy density of the thermal CFT.

- From $\langle \mathcal{O}^{\mu_1 \cdots \mu_J}(x) \rangle_\beta = \frac{b_\mathcal{O}}{\beta^\Delta}(\cdots)$ the energy density is given by

? Don't get where this comes from... unless we know $F = \frac{b_T}{d} T^d$ ⊛

$$\mathcal{E}(\beta) = -\langle T^{00} \rangle_\beta = -\left(1 - \frac{1}{d}\right) b_T \, \beta^{-d}$$

due to / requiring positivity of energy ≥ 0 $\therefore$ $b_T = -ve$

$\mathcal{Z}(\beta) = \mathrm{Tr}\, e^{-\beta H}$ $\therefore$ $E(\beta) = -\frac{\partial}{\partial \beta} \log \mathcal{Z} = \langle H \rangle_\beta$ in Euclidean $H = \int d^{d-1}x \; T^{00}(x)$

$\Rightarrow E(\beta) = \int d^{d-1}x \; \langle T^{00}(x) \rangle_\beta = \int d^{d-1}x \; \mathcal{E}(\beta)$ $\quad \mathcal{E}(\beta)$ is energy density

so: $\langle T^{00}(x) \rangle_\beta = \frac{b_T}{\beta^{\Delta_T}} x (\cdots)$

$\Delta_T = d$

For a CFT, $T_\mu^\mu = 0$. Thermal states preserve rot. so the ST takes the perfect fluid form

$\langle T^{\mu\nu} \rangle = \mathrm{diag}(\varepsilon, p, \cdots p)$

$T_\mu^\mu = -\varepsilon + (d-1)p$ $\quad \therefore \varepsilon = (d-1)p$, $p = -F$

- From dim-analysis, the free energy density takes the form:

$$F = f \beta^{-d}$$

where f is a dimensionless quantity $\quad \frac{dF}{dT} = f \frac{d}{dT} T^d = f \times d \times T^{d-1}$

- From thermodynamics: $F = E - TS = E + T \, dF/dT = E + f d T^d$

$$\varepsilon = -(d-1)F = -(d-1)(\varepsilon + f d T^d)$$

$$\rightarrow \varepsilon + (d-1)\varepsilon = -(d-1) f d T^d$$

$$\varepsilon = -\frac{(d-1)}{d} f d T^d \qquad \text{we know}$$

$$= -\left(1 - \frac{1}{d}\right) f d T^d$$

Allowed tensor structure

$\langle T^{\mu\nu} \rangle \sim (\varepsilon + p) u^\mu u^\nu + p \eta^{\mu\nu}$

$u^\mu = (1, 0, 0, \ldots)$ rest frame vector

$\therefore$

$\langle T^{\mu\nu} \rangle \sim \left(\varepsilon + \frac{\varepsilon}{d-1}\right) u^\mu u^\nu + \frac{\varepsilon}{d-1} \eta^{\mu\nu}$

$\hookrightarrow \eta^{00} = -1$!

- $\langle T^{00} \rangle_\beta = \varepsilon \, b_T T^d = -\left(1 - \frac{1}{d}\right) f \, d T^d$ ⊛

- The Ward Identity fixes:

$$⊛ \quad f_{\phi\phi T} = -\frac{d}{d-1} \frac{\Delta_\phi}{S_d} \qquad S_d = \mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

- The co-efficient of T in the thermal block expansion of $\langle \phi \phi \rangle_\beta$ is

$$a_T = \frac{f_{\phi\phi T} \, b_T}{C_T} \frac{J!}{2^J (\nu)_J}$$

→ Plug in ⊛

$$\Rightarrow a_T = -f S_d \frac{2\Delta_\phi}{d-2} \frac{C_{free}}{C_T} \quad \text{where } C_{free} = \frac{d}{d-1} \frac{1}{S_d^2}$$

$T^{\mu\nu}$ 2pt. coeff. for free boson in d dim.

- For a single real scalar

$$b_T = -2d\,\zeta(d)/S_d$$

$$F = f T^d = T \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \log\left[1 - e^{-\beta|k|}\right] = -\frac{2}{S_d}\zeta(d) T^d$$

- Some known results!

1. For the free scalar in three dimensions, we have $b_T^{\text{free}} = -6\zeta(3)/(4\pi) \approx -0.57394$.
2. For the $O(N)$ model in three dimensions at leading order in $1/N$, $b_T = 4N/5 \times b_T^{\text{free}} \approx -0.45915N$ [57, 48]. We will derive this from our inversion formula in section 5.1.
3. In the Monte Carlo literature, the quantity f is known as the "Casimir Amplitude". For the Ising model, Monte Carlo results give $f \approx -0.153$ [58–60], with numerical errors in the third digit. This translates to $b_T^{\text{Ising}} \approx -0.459$. Note that b_T^{Ising} is remarkably close to the value of b_T/N for the $O(N)$ model at large N.[14]