The Weil Conjectures (intro) - Ander

Statements Fix an alg. closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Z}}$ of $\mathbb{Z}$

Theorem (Weil Conjectures)

(i) Let X be a scheme of finite type over $\mathbb{F}_q$. Then, there exist $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in \overline{\mathbb{Z}}$ such that

$$\# X(\mathbb{F}_{q^n}) = (\alpha_1^n + \dots + \alpha_r^n) - (\beta_1^n + \dots + \beta_s^n)$$

for all $n \geq 1$

(ii) If X is a smooth proper variety of dim $= d$ over $\mathbb{F}_q$, then we can group the terms above as:

$$\# X(\mathbb{F}_{q^n}) = \sum_{j=1}^{b_0} \alpha_{0j}^n - \sum_{j=1}^{b_1} \alpha_{1j}^n + \sum_{j=1}^{b_2} \alpha_{2j}^n - \dots$$

$$+\sum_{j=1}^{b_{2d}} \alpha_{2d,j}^{n} \quad , \text{ where}$$

- The $b_i \in \mathbb{N}$ are called the ℓ-adic Betti numbers, and they satisfy $b_{2d-i} = b_i$ for $i = 0, \ldots, 2d$
- $\alpha_{ij} \in \overline{\mathbb{Z}}$ are such that $\alpha_{2d-i,*}$ equals $q^d / \alpha_{i,*}$

2d-i batch

i-th batch

- $|\alpha_{ij}| = q^{i/2}$ for all i, j and any archimedean absolute value on $\mathbb{Q}(\alpha_{ij})$

This is called Riemann Hypothesis for X

(iii) Let X be a proper smooth scheme over a finitely generated subring R of $\mathbb{C}$. Let m be a maximal ideal of R, and consider the reduction

typically $\mathcal{O}_{K,S}$ for a number field

$X_{R/m}$. Then, for $i = 0, \ldots, 2d$, the b_i in (ii) for $X_{R/m}$ are equal to rank $H^i(X(\mathbb{C}), \mathbb{Z})$

For curves: If X is a nice curve of genus g over $\mathbb{C}$, then

$$H^0(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}, \quad H^1(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^{2g},$$

$$H^2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$$

If X is a nice curve of genus g over $\mathbb{F}_q$, then the ℓ-adic Betti numbers are

$b_0 = 1, \quad b_1 = 2g, \quad b_2 = 1$

For curves, the Weil Conjectures tell us that $\exists\, \lambda_1, \ldots, \lambda_{2g} \in \overline{\mathbb{Z}}$ with $|\lambda_j| = q^{1/2}$ and $\lambda_{g+i} = \frac{q}{\lambda_i}$ for $i = 1, \ldots, g$ such that, for all $n \in \mathbb{N}$:

$$\# X(\mathbb{F}_{q^n}) = 1 - (\lambda_1^n + \ldots + \lambda_{2g}^n) + q^n$$

Corollary (Hasse–Weil bound) For a nice curve of genus g over $\mathbb{F}_q$, we have

$$\# X(\mathbb{F}_q) = q + 1 - \varepsilon \quad , \quad \text{with } |\varepsilon| \leq 2g\sqrt{q}$$

Def The Riemann zeta function is the meromorphic continuation of the holomorphic function defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$

Prop $$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

$$= \prod_{\substack{\text{maximal ideals} \\ m \subseteq \mathbb{Z}}} \left(1 - \#(\mathbb{Z}/m)^{-s}\right)^{-1} =$$

$$= \prod_{\substack{\text{closed points} \\ P \in \operatorname{Spec} \mathbb{Z}}} \left(1 - \# K(P)^{-s}\right)^{-1}$$

Def For a scheme X of finite type over $\mathbb{Z}$, we define the zeta function of X as

$$\zeta_X(s) := \prod_{\substack{\text{closed} \\ p \in X}} \left(1 - \# k(p)^{-s}\right)^{-1}$$

Remark ζ_X will converge on $\{ s \in \mathbb{C} : \operatorname{Re}(s) > r \}$ for some r depending on X.

Def Let X be a scheme of finite type over $\mathbb{F}_q$. Define

$$Z_X(T) := \exp\left(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right) \in$$

$$\in \mathbb{Q}[[T]]$$

$$Z_X(0) = 1, \quad T \frac{d}{dT} \log Z_X(T) = \sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n}) \cdot T^n$$

Prop If X is of finite type over $\mathbb{F}_q$, then it is also of finite type over $\mathbb{Z}$, and we have $\zeta_X(s) = Z_X(q^{-s})$.

$\rightsquigarrow$ Z_X is also called the zeta function of X.

<u>Theorem</u> (Weil Conjectures, restated)

(i) Let $X/\mathbb{F}_q$ scheme of finite type. Then the power series $Z_X(T)$ is the Taylor series of a rational function in $\mathbb{Q}(T)$. The rational function will be of the form

$$\frac{(1-\beta_1 T)\cdots(1-\beta_s T)}{(1-\alpha_1 T)\cdots(1-\alpha_r T)} \quad \text{for some}$$

$\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \in \overline{\mathbb{Z}}$

(ii) If $X/\mathbb{F}_q$ is a smooth, proper variety of $\dim = d$, then

$$Z_X(T) = \frac{P_1(T)P_3(T)\dots P_{2d-1}(T)}{P_0(T)P_2(T)\dots P_{2d}(T)}, \text{ where}$$

$P_i \in 1 + T\mathbb{Z}[T]$ factors over $\mathbb{C}$ as

$\prod_{j=1}^{b_i}(1-\alpha_{ij}T)$, with $|\alpha_{ij}| = q^{i/2}$. Also,

we have the equation

$$Z_X\left(\frac{1}{q^d T}\right) = \pm q^{d\cdot\chi/2} T^{\chi} \cdot Z_X(T)$$

$\chi := b_0 - b_1 + b_2 - \dots + b_{2d} \in \mathbb{Z}$

$\Uparrow$

the Euler characteristic of X

(iii) Same as before

Remark If X is a smooth curve over $\mathbb{F}_q$,

then the zeros of $Z_X(T)$ satisfy

$|T| = q^{-1/2}$, so the zeros of $\zeta_X(s)$ satisfy

$\mathrm{Re}(s) = 1/2$

Cohomological explanation The problem of computing $\#X(\mathbb{F}_{q^n})$ can be stated purely geometrically. Those points will be fixed by F^n ← the n-th power of (relative) Frobenius.

This is equal to the intersection number of two curves on $X \times X$, given by the graphs of the identity and F^n.

Lefschetz trace formula

- Topology: Given X compact differentiable manifold of $\dim X = d$ and $f: X \to X$ differentiable, a fixed point is $x \in X$ s.t. $f(x) = x$. At fixed points, $df_x \in \mathrm{End}(T_x X)$, and we call a fixed point non-degenerate if $I - df_x$ is invertible (as an endomorphism). "Fixed point is of multiplicity one"

For $i \geq 0$, $H^i(X, \mathbb{Z})$ is a f.g. abelian group, so $H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a f.d. vector space, isomorphic to $H^i(X, \mathbb{Q})$, and $b_i := \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$ is called the i-th Betti number of X. f induces an endomorphism f^* on $H^i(X, \mathbb{Q})$. The trace of f^* is an integer, because

$$\mathrm{Hom}(H^i(X,\mathbb{Q}), H^i(X,\mathbb{Q})) =$$

$$= \mathrm{Hom}(H^i(X,\mathbb{Z}) \otimes \mathbb{Q}, H^i(X,\mathbb{Z}) \otimes \mathbb{Q}) \simeq$$

$$\simeq \mathrm{Hom}(H^i(X,\mathbb{Z}), \mathrm{Hom}(H^i(X,\mathbb{Z}) \otimes \mathbb{Q}, \mathbb{Q}))$$

$$\simeq \mathrm{Hom}(H^i(X,\mathbb{Z}), \mathrm{Hom}(H^i(X,\mathbb{Z}), \mathbb{Q})) \simeq$$

$$\simeq \mathrm{Hom}(H^i(X,\mathbb{Z}), H^i(X,\mathbb{Z})) \otimes \mathbb{Q}$$

not canonical if $H^i(X,\mathbb{Z})$ not torsion-free

Theorem (Lefschetz trace formula) for compact manifolds If all fixed points of f are non-degenerate, then

$$\#\{\text{fixed points of } f\} = $$

$$= \sum_{i \geq 0} (-1)^i \operatorname{tr}\left(f^* \mid H^i(X, \mathbb{Q})\right)$$

$H^i(X, \mathbb{Q}) = 0$ for $i > d = \dim X$

$\Rightarrow$ the sum is finite

Notation $\operatorname{tr}(f \mid H^*(X, \mathbb{Q})) :=$

$$= \sum_{i \geq 0} (-1)^i \operatorname{tr}(f^* \mid H^i(X, \mathbb{Q}))$$

Some variable

$$\det(1 - Tf \mid H^*(X, \mathbb{Q})) =$$

$$= \prod_{i \geq 0} \det\left(1 - T \cdot f^* \mid H^i(X, \mathbb{Q})\right)^{(-1)^i}$$

manifold

Thm (Poincaré Duality) If X is an oriented, compact, real differentiable manifold of $\dim X = d$, then $H^d(X, \mathbb{Q}) \simeq \mathbb{Q}$, and there are pairings (given by the cup product)

$$H^i(X, \mathbb{Q}) \times H^{d-i}(X, \mathbb{Q}) \to H^d(X, \mathbb{Q}) \simeq \\ \simeq \mathbb{Q}$$

that are perfect pairings

In particular, $b_i = b_{d-i}$

Rmk If X is a complex manifold of dim d, then it's automatically oriented, and of dim $2d$ over $\mathbb{R}$

Def Let X be a scheme. Fix a prime

ℓ with $1/\ell \in \mathcal{O}_X$. For $i \in \mathbb{N}$, we define

$$H^i(X, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\acute{e}t}(X, \mathbb{Z}/\ell^n\mathbb{Z})$$

ℓ-adic cohomology

we will skip the subscript ét

$$H^i(X, \mathbb{Q}_\ell) := H^i(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

<u>Example</u> Let $A_{/K}$ be an abelian variety of dimension g. The Tate module of A is

let ℓ be a prime, $\ell \nmid \mathrm{char}\, K$

$$T_\ell A := \varprojlim_n \overbrace{A(K_s)[\ell^n]}^{(\mathbb{Z}/\ell^n\mathbb{Z})^2} =$$

came with a Galois action

$$= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A(K_s)) \simeq$$

$$\simeq \mathbb{Z}_\ell^{2g} \simeq H_1^{\acute{e}t}(A, \mathbb{Z}_\ell)$$

$$V_\ell A = T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^{2g}$$

<u>Rmk</u> The ℓ-adic Betti number b_i is also equal to $\dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell)$

<u>Warning</u> If k is not separably closed, then the Betti numbers of X are defined to be those of $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k^s$

<u>Def</u> Let X be a scheme, let $n \in \mathbb{N}$ s.t. $\frac{1}{n} \in \mathcal{O}_X$. For $m \in \mathbb{Z}$, the Tate twist $(\mathbb{Z}/n\mathbb{Z})(m)$ is a sheaf on $X_{\acute{e}t}$ as follows:

$$\mathbb{Z}/n\mathbb{Z}(m) := \begin{cases} \mathbb{Z}/n\mathbb{Z}, & m = 0 \\ (\mu_n)^{\otimes m}, & m > 0 \\ \operatorname{Hom}(\mu_n^{\otimes -m}, \mathbb{Z}/n\mathbb{Z}), & m < 0 \end{cases}$$

Fix prime ℓ with $\frac{1}{\ell} \in \mathcal{O}_X$ and $m \in \mathbb{Z}$

For each $n \geq 0$, we have a natural map

$\mathbb{Z}/\ell^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/\ell^n\mathbb{Z}$. We define

$$H^i(X, \mathbb{Z}_\ell(m)) := \varprojlim_n H^i_{ét}(X, \mathbb{Z}/\ell^n\mathbb{Z}(m))$$

$$H^i(X, \mathbb{Q}_\ell(m)) := H^i(X, \mathbb{Z}_\ell(m)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

$\mathbb{Q}_\ell(1)$ is called cyclotomic character

<u>Prop</u> $H^i(X_{K_s}, \mathbb{Q}_\ell(m)) \simeq$ of $\mathbb{Q}_\ell$-representations of $G_K = \mathrm{Gal}(K_s/K)$

$$\simeq H^i(X_{K_s}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(m)$$

<u>Proof</u> Over K_s, $\mathbb{Z}/\ell^n\mathbb{Z}(m)$ can be

identified with $\mathbb{Z}/\ell^n\mathbb{Z}$ by choosing a

generator ζ of $\mathbb{Z}/\ell^n\mathbb{Z}(m)$ (as G_n-module)

$$\Rightarrow H^i(X_{ks}, \mathbb{Z}/\ell^n\mathbb{Z}) \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\otimes} (\mathbb{Z}/\ell^n\mathbb{Z})(m) \to$$

$$\to H^i(X_{ks}, \mathbb{Z}/\ell^n\mathbb{Z}(m)).$$ This iso

is independent of the choice of $\zeta \Rightarrow$

$\Rightarrow$ it's G_n-equivariant $\Rightarrow$ take inverse

limits and $-\otimes \mathbb{Q}_\ell$ $\square$

<u>Thm</u> (Poincaré Duality in ℓ-adic

cohomology). Let X/k smooth proper

integral variety of $\dim X = d$, k separably

closed. Fix $\ell \neq \operatorname{char} k$

(a) $\exists$ natural iso $H^{2d}(X, \mathbb{Q}_\ell(d)) \simeq \mathbb{Q}_\ell$

(b) The cup product gives a perfect pairing

$$H^r(X, \mathbb{Q}_\ell(i)) \times H^{2d-r}(X, \mathbb{Q}_\ell(d-i)) \to$$

$$\to H^{2d}(X, \mathbb{Q}_\ell(d)) \cong \mathbb{Q}_\ell$$

$\forall r, i \in \mathbb{Z}$

<u>Deducing Weil conjectures</u> $X/\mathbb{F}_q$ smooth proper variety of dimension d. $\overline{X} = X_{\overline{\mathbb{F}}_q}$

$F: \overline{X} \to \overline{X}$ relative Frobenius.

Then

$$\# X(\mathbb{F}_{q^n}) = \operatorname{tr}(F^n \mid H^*(\overline{X}, \mathbb{Q}_\ell))$$

Thus

$$Z_X(T) = \exp\left(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n}) \frac{T^n}{n}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \operatorname{tr}(F^n \mid H^*(\overline{X}, \mathbb{Q}_\ell)) \frac{T^n}{n}\right)$$

$$= \det\left(1 - TF \mid H^*(\overline{X}, \mathbb{Q}_\ell)\right)^{-1}$$

$$= \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}, \text{ where}$$

$$P_i(t) = \det\left(1 - TF \mid H^i(\overline{X}, \mathbb{Q}_\ell)\right) \in$$
$$\in \mathbb{Q}_\ell[T]$$

$\Rightarrow Z_X(T)$ is a rational function in $\mathbb{Q}_\ell(T)$. But $Z_X(T)$ is also in $1 + T\mathbb{Z}[[T]] \Rightarrow Z_X \in \mathbb{Q}(T)$.

Let $b_i = \dim H^i(\overline{X}, \mathbb{Q}_\ell)$, let $\alpha_{i1}, \dots, \alpha_{i,b_i}$ be the eigenvalues of $F^* \mid H^i(\overline{X}, \mathbb{Q}_\ell)$ (w/ multiplicity)

$$\Rightarrow P_i(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} T)$$

$$\alpha_{ij} \neq 0 \Rightarrow \deg P_i = b_i$$

If $\overline{X}$ is integral, then

$$H^{2d}(\overline{X}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-d)$$

$$(\Rightarrow b_{2d} = 1)$$

F^* acts on $H^{2d}(\overline{X}, \mathbb{Q}_\ell)$ as multiplication by q^d. The eigenvalues $\alpha_{2d-i,*}$ of F^* acting on $H^{2d-i}(\overline{X}, \mathbb{Q}_\ell)$ are the inverses of those of F^* acting on $H^i(\overline{X}, \mathbb{Q}_\ell(d))$, which are equal to

$\frac{\alpha_i}{q^d}$ because of the twist.

$\Rightarrow$ we get the functional equation.

Remains to prove: $|\alpha_{ij}| = q^{i/2}$, $\alpha_{ij} \in \overline{\mathbb{Z}}$

(shown by Deligne using ℓ-adic cohom.)