# Recap

**def** for $p \in P_n(A)$, $q \in P_m(A)$, $p \sim_0 q$ in $P_\infty(A)$ if $\exists v \in M_{m,n}(A)$ s.t. $p = v^*v$, $q = vv^*$

**def** $\oplus : P_\infty(A) \times P_\infty(A) \to P_\infty(A)$ via $p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$

**prop**
1) $p \sim_0 p \oplus 0$
2) $p \sim_0 p'$, $q \sim_0 q' \Rightarrow p \oplus q \sim_0 p' \oplus q'$
3) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$
4) $p \oplus q \sim_0 q \oplus p$
5) $pq = 0 \Rightarrow p + q \sim_0 p \oplus q$

**def** by previous prop., we can define $\mathcal{D}(A) = P_\infty(A)/\sim_0$, which is an abelian semigroup with addition given by $[p] + [q] = [p \oplus q]$ [unital]

## Grothendieck construction

**def** let $S$ be an abelian semigroup and let $\Delta : S \to S \times S$ be the diagonal embedding $x \mapsto (x, x)$. define the Grothendieck group of $S$ to be $G(S) = S \times S / \Delta(S)$; this inherits the abelian semigroup structure from $S \times S$, and moreover, the inverse of $(x, y)$ in $G(S)$ is $(y, x)$, so $G(S)$ is a group. also $\alpha : S \to G(S)$, $x \mapsto (x, 0)$ is the Grothendieck map

**prop**
1) $G(S)$ has the universal property: given $H$ a group and $\varphi : S \to H$ a semigroup morphism, the diagram on the right commutes

$$\begin{tikzcd} S \arrow[d, "\alpha"'] \arrow[dr, "\varphi"] & \\ G(S) \arrow[r, dashed, "\exists!\psi"'] & H \end{tikzcd}$$

2) $G$ is a functor

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & T \arrow[d, "\alpha"] \\ G(S) \arrow[r, "G(\varphi)"'] & G(T) \end{tikzcd}$$

3) $G(S) = \{\alpha(x) - \alpha(y) \mid x, y \in S\}$
4) $\alpha(x) = \alpha(y) \Leftrightarrow \exists z \in S$ s.t. $x + z = y + z$
5) $\alpha$ injective $\Leftrightarrow$ $S$ has cancellation property

6) let $H$ be an abelian group. $0 \neq S \subset H$, if $S$ is closed under addition, then $S$ has cancellation property, $G(S) \cong \langle S \rangle < G$, and $\langle S \rangle = \{x - y \mid x, y \in S\}$

ex 1) if $S = \mathbb{Z}^+$, then $G(S) \cong \mathbb{Z}$

2) if $S = \mathbb{Z}^+ \cup \{\infty\}$, then $G(S) = 1$

def let $A$ be a unital $C^*$-algebra. define $K_0(A) = G(\mathcal{D}(A))$ and $[-]_0 : P_\infty(A) \to K_0(A)$ by $[p]_0 = \alpha([p])$

def the definition of $K_0(A)$ can be extended to non-unital $C^*$-algebras word for word; this group is denoted $K_{00}(A)$ (the actual extension of $K_0$ to non-unital $C^*$-algebras is different)

def define a relation on $P_\infty(A)$: $p \sim_s q$ iff $p \oplus r \sim_0 q \oplus r$ for some $r \in P_\infty(A)$, this is called stable equivalence

prop $p \sim_s q \iff \exists n \in \mathbb{Z}^+$ s.t. $p \oplus 1_n \sim_0 q \oplus 1_n$, $1_n \in M_n(A)$ identity matrix

proof $p \oplus 1 \sim_0 p \oplus r \oplus (1-r) \sim_0 q \oplus r \oplus (1-r) \sim_0 q \oplus 1$

prop $A$ unital $C^*$-algebra, then

1) $K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in P_\infty(A)\} = \{[p]_0 - [q]_0 \mid p, q \in P_n(A), n \geq 1\}$

2) $[0]_0 = 0$

3) $[p \oplus q]_0 = [p]_0 + [q]_0$

4) $p, q \in P_n(A)$ and $p \sim_h q$ in $P_n(A)$ then $[p]_0 = [q]_0$

5) $pq = 0 \Rightarrow [p+q]_0 = [p]_0 + [q]_0$

6) $[p]_0 = [q]_0 \iff p \sim_s q$

proof 6) $[p]_0 = [q]_0 \iff \exists r \in P_\infty(A)$ s.t. $[p] + [r] = [q] + [r]$

$\iff [p \oplus r] = [q \oplus r]$

$\iff p \oplus r \sim_0 q \oplus r$

$\iff p \sim_s q$

**Prop** let $G$ be an abelian group, $\mu : P_\infty(A) \to G$ s.t.

1) $\mu(p \oplus q) = \mu(p) + \mu(q)$
2) $\mu(0) = 0$
3) $p \sim_h q$ in $P_n(A) \Rightarrow \mu(p) = \mu(q)$

$$\begin{tikzcd} P_\infty(A) \arrow[d, "{[-]_0}"'] \arrow[dr, "\mu"] & \\ K_0(A) \arrow[r, dashed, "!\varphi"'] & G \end{tikzcd}$$

then the following diagram commutes

**proof** Γassume $p \sim_0 q$,
$p \sim_0 p \oplus 0$ so wlog $p \sim q$, then $p \oplus 0_n \sim_u q \oplus 0_n \Rightarrow p \oplus 0_{3n} \sim_h q \oplus 0_{3n}$,
hence $\mu(p) = \mu(p) + 3n \cdot \mu(0) = \mu(p \oplus 0_{3n}) = \mu(q \oplus 0_{3n}) = \mu(q)$, so $\mu$ factorises through $\mathcal{D}(A)$, then apply universality of $G(\mathcal{D}(A)) = K_0(A)$

## $K_0$ is a functor

Γ Category theory background: a category $\mathcal{C}$ consists of a
- collection of objects $obj(\mathcal{C})$, and
- collection of arrows $mor(\mathcal{C})$,

such that
- for every pair of arrows $f : X \to Y$, $g : Y \to Z$, there's an arrow $g \circ f : X \to Z$ (composition)
- for every object $X$ there's an arrow $1_X$ which is the identity wrt composition (i.e. for every arrow $f : X \to Y$, $f 1_X = f = 1_Y f$)

a functor $F : \mathcal{C} \to \mathcal{D}$ between categories is a structure preserving map, that is,
- for every $X \in obj(\mathcal{C})$, assign $FX \in obj(\mathcal{D})$
- for every arrow $f : X \to Y$ in $\mathcal{C}$, an arrow $Ff : FX \to FY$ in $\mathcal{D}$
- $F(1_X) = 1_{FX}$
- $F(g \circ f) = Fg \circ Ff$

a contravariant functor is a functor that flips the direction of arrows an object $X \in obj(\mathcal{C})$ is a zero object if for every $Y \in obj(\mathcal{C})$, $Hom(X,Y)$ and $Hom(Y,X)$ contain exactly one arrow each ┘

let $\varphi: A \to B$ be a $*$-hom, then it extends to $\varphi: M_n(A) \to M_n(B)$, and further restricts to $\varphi: P_n(A) \to P_n(B)$. combining all together we get a map $\varphi: P_\infty(A) \to P_\infty(B)$. define $\mu: P_\infty(A) \to K_0(B)$ by $\mu(p) = [\varphi(p)]_0$, then by universality, $\mu$ factors through $K_0(A)$, giving a map which we denote $K_0(\varphi): K_0(A) \to K_0(B)$, given by

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0$$

prop 1) $K_0(id_A) = id_{K_0(A)}$

2) $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$

3) $K_0(0) = 0$

4) $K_0(0_{B,A}) = 0_{K_0(B), K_0(A)}$

def $\varphi, \psi: A \to B$ are homotopic, written $\varphi \sim_h \psi$, if there is a family of $*$-homs $\varphi_t: A \to B$ s.t. $\varphi_0 = \varphi$, $\varphi_1 = \psi$, and $\varphi_t(a): [0,1] \to B$ is cts $\forall a \in A$; we also say $A, B$ are homotopy equivalent if $\exists\, \varphi: A \to B$, $\psi: B \to A$ $*$-homs s.t. $\psi \circ \varphi \sim_h id_A$ and $\varphi \circ \psi \sim_h id_B$

prop 1) $\varphi, \psi: A \to B$ homotopic $\Rightarrow K_0(\varphi) = K_0(\psi)$

2) $A, B$ homotopy equivalent $\Rightarrow K_0(A) \cong K_0(B)$

def $*$-homs $\varphi, \psi: A \to B$ are mutually orthogonal, written $\varphi \perp \psi$, if $\varphi(A)\psi(A) = 0$

lem $\varphi \perp \psi \Rightarrow \varphi + \psi$ is a $*$-hom and $K_0(\varphi+\psi) = K_0(\varphi) + K_0(\psi)$

proof note that $\varphi_n, \psi_n: M_n(A) \to M_n(B)$ are orthogonal and $(\varphi+\psi)_n = \varphi_n + \psi_n$, so we get

$$\begin{aligned} K_0(\varphi+\psi)([p]_0) &= [(\varphi+\psi)(p)]_0 \\ &= [\varphi(p) + \psi(p)]_0 \\ &= [\varphi(p)]_0 + [\psi(p)]_0 \qquad \text{since } \varphi(p)\psi(p) = 0 \\ &= (K_0(\varphi) + K_0(\psi))([p]_0) \end{aligned}$$

□

**Rem** for $A$ unital $C^*$-algebra, the split exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

induces a split exact sequence

$$0 \longrightarrow K_0(A) \xrightarrow{K_0(\iota)} K_0(\tilde{A}) \underset{K_0(\lambda)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(\mathbb{C}) \longrightarrow 0$$

**Proof** let $f = 1_{\tilde{A}} - 1_A$, then $\tilde{A} = A \oplus \mathbb{C}f$, and define $\mu : \tilde{A} \to A$, $\lambda' : \mathbb{C} \to \tilde{A}$ by

$$\mu(a + \alpha f) = a, \quad \lambda'(\alpha) = \alpha f.$$

then

$$id_A = \mu \circ \iota, \quad id_{\tilde{A}} = \iota \circ \mu + \lambda' \circ \pi, \quad \pi \circ \iota = 0, \quad \pi \circ \lambda = id_{\mathbb{C}},$$

and $\iota \circ \mu \perp \lambda' \circ \pi$. then by functoriality:

(i) $K_0(\pi) \circ K_0(\iota) = K_0(\pi \circ \iota) = K_0(0) = 0$,

(ii) $K_0(\pi) \circ K_0(\lambda) = K_0(\pi \circ \lambda) = id_{K_0(\mathbb{C})}$,

(iii) $K_0(\mu) \circ K_0(\iota) = K_0(\mu \circ \iota) = id_{K_0(A)}$,

(iv) $K_0(\iota) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi) = id_{K_0(\tilde{A})}$.

then (ii) $\Rightarrow$ $K_0(\pi)$ split + surjective, (iii) $\Rightarrow$ $K_0(\iota)$ injective, (i)+(iv) $\Rightarrow$ ~~split~~ exact at $K_0(\tilde{A})$ $\square$

## Examples

**ex** consider the trace map $Tr : M_n(\mathbb{C}) \to \mathbb{C}$ and note that it's a positive trace, i.e. $Tr(a) \geq 0$ for $a \in M_n(\mathbb{C})$ positive and $Tr(ab) = Tr(ba)$; this induces a unique trace $Tr_K : M_K(A) \to \mathbb{C}$, where $A = M_n(\mathbb{C})$, s.t. $Tr_K(diag(a, 0, \dots, 0)) = Tr(a)$, hence we get a map $Tr : P_\infty(A) \to \mathbb{C}$, which by universality induces a map $K_0(Tr)([p]_0) = Tr(p)$.

let $g \in K_0(A)$, then $\exists p, q \in P_K(A)$ s.t. $g = [p]_0 - [q]_0$, and so

$$K_0(Tr)(g) = Tr(p) - Tr(q) = rank(p) - rank(q),$$

hence $K_0(Tr)(g) \in \mathbb{Z}$. in particular, if $K_0(Tr)(g) = 0$, then $p \sim q$, and so $g = 0$, i.e. $K_0(Tr) : K_0(A) \to \mathbb{Z}$ is injective. finally, for any rank one projection $e \in A$, $K_0(Tr)([e]_0) = 1$, so $K_0(Tr)$ is an isomorphism.

**ex** let $H$ be an infinite-dimensional separable Hilbert space and identify $M_n(B(H)) = B(H^n)$. we then get a surjective map $\dim: P_\infty(B(H)) \to \mathbb{Z}^+ \cup \{\infty\}$. note that for $p, q \in P(B(H^n))$, $\dim p = \dim q$ iff $p \sim q$, hence for any $p, q \in P_\infty(B(H))$, $\dim p = \dim q$ iff $p \sim_0 q$. since dimension is additive, $\dim(p \oplus q) = \dim(p) + \dim(q)$. hence $\dim$ induces an isomorphism $\mathcal{D}(B(H)) \to \mathbb{Z}^+ \cup \{\infty\}$, and so $K_0(B(H)) \cong G(\mathbb{Z}^+ \cup \{\infty\}) = 0$.

* also true when $H$ is not separable

**ex** let $X$ be a cpct, connected Hdff space, $A = C(X)$. then for $p \in P_n(A)$, $p(x) \in M_n(\mathbb{C})$ $\forall x \in X$. note that $x \mapsto Tr(p(x))$ is in $C(X, \mathbb{Z})$, and since $X$ is connected, this map is constant, i.e. $Tr(p(x))$ is independent of $x \in X$. let $\tau_x : C(X) \to \mathbb{C}$ be given by $\tau_x(f) = f(x)$. then $\tau_x$ is a trace, hence we get a map $K_0(\tau_x) : K_0(A) \to \mathbb{C}$. by uniqueness of trace, $K_0(\tau_x)([p]_0) = \tau_x(p) = Tr(p(x))$, hence $K_0(\tau_x)$ is independent of $x \in X$ and its image is in $\mathbb{Z}$. finally, $1 = 1_A(x) = K_0(\tau_x)([1_A]_0)$, hence $K_0(\tau_x) : K_0(A) \to \mathbb{Z}$ is surjective.

**ex** let $X$ be as before + contractible, i.e. $\exists x_0 \in X$, $\alpha : X \times I \xrightarrow{\text{cts}} X$ s.t. $\alpha(1, x) = x$, $\alpha(0, x) = x_0$. for $t \in [0,1]$ define $\varphi_t : C(X) \to C(X)$ given by $\varphi_t(f)(x) = f(\alpha(t, x))$. then $\varphi_0(f)(x) = f(x_0)$ and $\varphi_1 = id$; moreover, $t \mapsto \varphi_t(f)$ is cts, hence $\varphi_0 \sim_h id$.

now let $\varphi : C(X) \to \mathbb{C}$, $\psi : \mathbb{C} \to C(X)$ be given by $\varphi(f) = f(x_0)$, $\psi(\lambda) = \lambda \cdot 1$.

then $\varphi \circ \psi = id_{\mathbb{C}}$ and $\psi \circ \varphi = \varphi_0 \sim_h id$. by homotopy ~~equivalence~~ invariance, $K_0(\varphi)$ is an isomorphism. finally, the commutativity of the diagram on the right implies $K_0(\tau_x)$ is a composition of isomorphisms, hence an isomorphism itself.

$$\begin{tikzcd} K_0(A) \arrow[d, "K_0(\varphi)"'] \arrow[dr, "K_0(\tau_x)"] & \\ K_0(\mathbb{C}) \arrow[r, "K_0(Tr)"'] & \mathbb{Z} \end{tikzcd}$$