

## Recap

①

def for  $p \in P_n(A)$ ,  $q \in P_m(A)$ ,  $p \sim_0 q$  in  $P_\infty(A)$  if  $\exists v \in M_{m,n}(A)$  s.t.  
 $p = v^*v$ ,  $q = vv^*$

def  $\oplus: P_\infty(A) \times P_\infty(A) \rightarrow P_\infty(A)$  via  $p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$

- prop
- 1)  $p \sim_0 p \oplus 0$
  - 2)  $p \sim_0 p'$ ,  $q \sim_0 q' \Rightarrow p \oplus q \sim_0 p' \oplus q'$
  - 3)  $(p \oplus q) \oplus r = p \oplus (q \oplus r)$
  - 4)  $p \oplus q \sim_0 q \oplus p$
  - 5)  $pq = 0 \Rightarrow p + q \sim_0 p \oplus q$

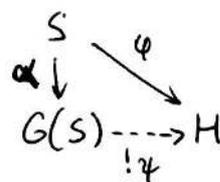
def by previous prop., we can define  $\mathcal{D}(A) = P_\infty(A) / \sim_0$ , which is an abelian semigroup with addition given by  $[p] + [q] = [p \oplus q]$

[unital]

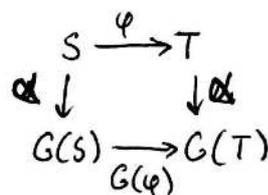
### Grothendieck construction

def let  $S$  be an abelian semigroup and let  $\Delta: S \rightarrow S \times S$  be the diagonal embedding  $x \mapsto (x, x)$ . define the Grothendieck group of  $S$  to be  $G(S) = S \times S / \Delta(S)$ ; this inherits the abelian semigroup structure from  $S \times S$ , and moreover, the inverse of  $(x, y)$  in  $G(S)$  is  $(y, x)$ , so  $G(S)$  is a group. also  $\alpha: S \rightarrow G(S)$  is the Grothendieck map  $x \mapsto (x, 0)$

prop 1)  $G(S)$  has the universal property:  
 given  $H$  a group and  $\varphi: S \rightarrow H$  a semigroup morphism, the diagram on the right commutes



2)  $G$  is a functor



3)  $G(S) = \{ \alpha(x) - \alpha(y) \mid x, y \in S \}$

4)  $\alpha(x) = \alpha(y) \Leftrightarrow \exists z \in S$  s.t.  $x+z = y+z$

5)  $\alpha$  injective  $\Leftrightarrow S$  has cancellation property

6) let  $H$  be an abelian group.  $0 \neq S \subset H$ , if  $S$  is closed under addition, then  $S$  has cancellation property,  $G(S) \cong \langle S \rangle \subset G$ , and  $\langle S \rangle = \{x-y \mid x, y \in S\}$

ex 1) if  $S = \mathbb{Z}^+$ , then  $G(S) \cong \mathbb{Z}$

2) if  $S = \mathbb{Z}^+ \cup \{\infty\}$ , then  $G(S) = 1$

def let  $A$  be a unital  $C^*$ -algebra. define  $K_0(A) = G(\mathcal{D}(A))$  and  $[-]_0: \mathcal{P}_\infty(A) \rightarrow K_0(A)$  by  $[p]_0 = \frac{1}{2} \alpha([p])$

def the definition of  $K_0(A)$  can be extended to non-unital  $C^*$ -algebras word for word; this group is denoted  $K_{00}(A)$  (the actual extension of  $K_0$  to non-unital  $C^*$ -algebras is different)

def define a relation on  $\mathcal{P}_\infty(A)$ :  $p \sim_s q$  iff  $p \oplus r \sim_0 q \oplus r$  for some  $r \in \mathcal{P}_\infty(A)$ , this is called stable equivalence

prop  $p \sim_s q \Leftrightarrow \exists n \in \mathbb{Z}^+$  s.t.  $p \oplus 1_n \sim_0 q \oplus 1_n$ ,  $1_n \in M_n(A)$  identity matrix

proof  $p \oplus 1 \sim_0 p \oplus r \oplus (1-r) \sim_0 p \oplus r \oplus (1-r) \sim_0 q \oplus 1$

prop A unital  $C^*$ -algebra, then

$$1) K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(A)\} = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(A), n \geq 1\}$$

$$2) [0]_0 = 0$$

$$3) [p \oplus q]_0 = [p]_0 + [q]_0$$

$$4) p, q \in \mathcal{P}_n(A) \text{ and } p \sim_h q \text{ in } \mathcal{P}_n(A) \text{ then } [p]_0 = [q]_0$$

$$5) pq = 0 \Rightarrow [p+q]_0 = [p]_0 + [q]_0$$

$$6) [p]_0 = [q]_0 \Leftrightarrow p \sim_s q$$

proof 6)  $[p]_0 = [q]_0 \Leftrightarrow \exists r \in \mathcal{P}_\infty(A)$  s.t.  $[p] + [r] = [q] + [r]$

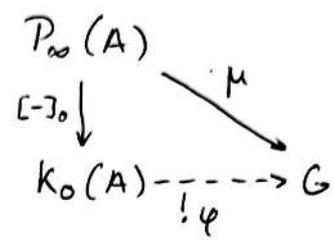
$$\Leftrightarrow [p \oplus r] = [q \oplus r]$$

$$\Leftrightarrow p \oplus r \sim_0 q \oplus r$$

$$\Leftrightarrow p \sim_s q$$

prop let  $G$  be an abelian group,  $\mu: P_{\infty}(A) \rightarrow G$  s.t.

- 1)  $\mu(p \oplus q) = \mu(p) + \mu(q)$
- 2)  $\mu(0) = 0$
- 3)  $p \sim_h q$  in  $P_n(A) \Rightarrow \mu(p) = \mu(q)$



then the following diagram commutes

Assume  $p \sim_0 q$ ,

proof  $p \sim_0 p \oplus 0$  so wlog  $p \sim q$ , then  $p \oplus 0_n \sim_u p \oplus 0_n \Rightarrow p \oplus 0_{3n} \sim_h q \oplus 0_{3n}$ ,  
 hence  $\mu(p) = \mu(p) + 3n \cdot \mu(0) = \mu(p \oplus 0_{3n}) = \mu(q \oplus 0_{3n}) = \mu(q)$ , so  $\mu$   
 factorises through  $\mathcal{D}(A)$ , then apply universality of  $G(\mathcal{D}(A)) = K_0(A)$

$K_0$  is a functor

Category theory background: a category  $\mathcal{C}$  consists of a

- collection of objects  $\text{obj}(\mathcal{C})$ , and
- collection of arrows  $\text{mor}(\mathcal{C})$ ,

such that

- for every pair of arrows  $f: X \rightarrow Y, g: Y \rightarrow Z$ , there's an arrow  $g \circ f: X \rightarrow Z$  (composition)
- for every object  $X$  there's an arrow  $1_X$  which is the identity wrt composition (i.e. for every arrow  $f: X \rightarrow Y$ ,  $f \circ 1_X = f = 1_Y \circ f$ )

a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  between categories is a structure preserving map, that is,

- for every  $X \in \text{obj}(\mathcal{C})$ , assign  $F X \in \text{obj}(\mathcal{D})$
- for every arrow  $f: X \rightarrow Y$  in  $\mathcal{C}$ , an arrow  $F f: F X \rightarrow F Y$  in  $\mathcal{D}$
- $F(1_X) = 1_{F X}$
- $F(g \circ f) = F g \circ F f$

a contravariant functor is a functor that flips the direction of arrows

an object  $X \in \text{obj}(\mathcal{C})$  is a zero object if for every  $Y \in \text{obj}(\mathcal{C})$ ,

$\text{Hom}(X, Y)$  and  $\text{Hom}(Y, X)$  contain exactly one arrow each

let  $\varphi: A \rightarrow B$  be a  $*$ -hom, then it extends to  $\varphi: M_n(A) \rightarrow M_n(B)$ , and further restricts to  $\varphi: P_n(A) \rightarrow P_n(B)$ . combining all together we

get a map  $\varphi: P_\infty(A) \rightarrow P_\infty(B)$ . define  $\mu: P_\infty(A) \rightarrow K_0(B)$  by  $\mu(p) = [\varphi(p)]_0$ , then by universality,  $\mu$  factors through  $K_0(A)$ , giving a map which we denote  $K_0(\varphi): K_0(A) \rightarrow K_0(B)$ , given by

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0$$

prop 1)  $K_0(\text{id}_A) = \text{id}_{K_0(A)}$

2)  $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$

3)  $K_0(0) = 0$

4)  $K_0(O_{B,A}) = O_{K_0(B), K_0(A)}$

def  $\varphi, \psi: A \rightarrow B$  are homotopic, written  $\varphi \sim_h \psi$ , if there is a family of  $*$ -homs  $\varphi_t: A \rightarrow B$  s.t.  $\varphi_0 = \varphi$ ,  $\varphi_1 = \psi$ , and  $\varphi_t(a): [0,1] \rightarrow B$  is cts  $\forall a \in A$ ; we also say  $A, B$  are homotopy equivalent if  $\exists \varphi: A \rightarrow B$ ,  $\psi: B \rightarrow A$   $*$ -homs s.t.  $\psi \circ \varphi \sim_h \text{id}_A$  and  $\varphi \circ \psi \sim_h \text{id}_B$

prop 1)  $\varphi, \psi: A \rightarrow B$  homotopic  $\Rightarrow K_0(\varphi) = K_0(\psi)$

2)  $A, B$  homotopy equivalent  $\Rightarrow K_0(A) \cong K_0(B)$

def  $*$ -homs  $\varphi, \psi: A \rightarrow B$  are mutually orthogonal, written  $\varphi \perp \psi$ , if  $\varphi(A)\psi(A) = 0$

lem  $\varphi \perp \psi \Rightarrow \varphi + \psi$  is a  $*$ -hom and  $K_0(\varphi + \psi) = K_0(\varphi) + K_0(\psi)$

proof note that  $\varphi_n, \psi_n: M_n(A) \rightarrow M_n(B)$  are orthogonal and  $(\varphi + \psi)_n = \varphi_n + \psi_n$ , so we get

$$K_0(\varphi + \psi)([p]_0) = [(\varphi + \psi)(p)]_0$$

$$= [\varphi(p) + \psi(p)]_0$$

$$= [\varphi(p)]_0 + [\psi(p)]_0 \quad \text{since } \varphi(p)\psi(p) = 0$$

$$= (K_0(\varphi) + K_0(\psi))( [p]_0 )$$

□

lem for A unital C\*-algebra, the split exact sequence

$$0 \rightarrow A \xrightarrow{\iota} \tilde{A} \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{\lambda} \end{matrix} \mathbb{C} \rightarrow 0$$

induces a split exact sequence

$$0 \rightarrow K_0(A) \xrightarrow{K_0(\iota)} K_0(\tilde{A}) \begin{matrix} \xrightarrow{K_0(\pi)} \\ \xleftarrow{K_0(\lambda)} \end{matrix} \mathbb{Z} \rightarrow 0$$

proof let  $f = 1_{\tilde{A}} - 1_A$ , then  $\tilde{A} = A \oplus \mathbb{C}f$ , and define  $\mu: \tilde{A} \rightarrow A, \lambda': \mathbb{C} \rightarrow \tilde{A}$  by  $\mu(a + \alpha f) = a, \lambda'(\alpha) = \alpha f$ .

then

$$id_A = \mu \circ \iota, \quad id_{\tilde{A}} = \iota \circ \mu + \lambda' \circ \pi, \quad \pi \circ \iota = 0, \quad \pi \circ \lambda = id_{\mathbb{C}},$$

and  $\iota \circ \mu \perp \lambda' \circ \pi$ . then by functoriality:

- (i)  $K_0(\pi) \circ K_0(\iota) = K_0(\pi \circ \iota) = K_0(0) = 0,$
- (ii)  $K_0(\pi) \circ K_0(\lambda) = K_0(\pi \circ \lambda) = id_{K_0(\mathbb{C})},$
- (iii)  $K_0(\mu) \circ K_0(\iota) = K_0(\mu \circ \iota) = id_{K_0(A)},$
- (iv)  $K_0(\iota) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi) = id_{K_0(\tilde{A})}.$

then (ii)  $\Rightarrow K_0(\pi)$  split + surjective, (iii)  $\Rightarrow K_0(\iota)$  injective, (i)+(iv)  $\Rightarrow$  ~~split~~ exact at  $K_0(\tilde{A})$   $\square$

### Examples

ex consider the trace map  $Tr: M_n(\mathbb{C}) \rightarrow \mathbb{C}$  and note that it's a ~~trace~~ <sup>positive trace</sup>, i.e.  $Tr(a) \geq 0$  for  $a \in M_n(\mathbb{C})$  positive and  $Tr(ab) = Tr(ba)$ ; this induces a unique trace  $Tr_k: M_k(A) \rightarrow \mathbb{C}$ , where  $A = M_n(\mathbb{C})$ , s.t.  $Tr_k(\text{diag}(a, 0, \dots, 0)) = Tr(a)$ , hence we get a map  $Tr: P_{\infty}(A) \rightarrow \mathbb{C}$ , which by universality induces a map  $K_0(Tr)([p]_0) = Tr(p)$ .

let  $g \in K_0(A)$ , then  $\exists p, q \in P_k(A)$  s.t.  $g = [p]_0 - [q]_0$ , and so

$$K_0(Tr)(g) = Tr(p) - Tr(q) = \text{rank}(p) - \text{rank}(q),$$

hence  $K_0(Tr)(g) \in \mathbb{Z}$ . in particular, if  $K_0(Tr)(g) = 0$ , then  $p \sim q$ , and so  $g = 0$ , i.e.  $K_0(Tr): K_0(A) \rightarrow \mathbb{Z}$  is injective. finally, for any rank one projection  $e \in A, K_0(Tr)([e]_0) = 1$ , so  $K_0(Tr)$  is an isomorphism.

ex let  $H$  be an infinite-dimensional separable Hilbert space and identify  $M_n(B(H)) = B(H^n)$ . we then get a surjective map  $\dim: P_\infty(B(H)) \rightarrow \mathbb{Z}^+ \cup \{\infty\}$ . note that for  $p, q \in P(B(H^n))$ ,  $\dim p = \dim q$  iff  $p \sim q$ , hence for any  $p, q \in P_\infty(B(H))$ ,  $\dim p = \dim q$  iff  $p \sim_0 q$ . since dimension is additive,  $\dim(p \oplus q) = \dim(p) + \dim(q)$ . hence  $\dim$  induces an isomorphism  $\mathcal{D}(B(H)) \rightarrow \mathbb{Z}^+ \cup \{\infty\}$ , and so  $K_0(B(H)) \cong G(\mathbb{Z}^+ \cup \{\infty\}) = 0$ .

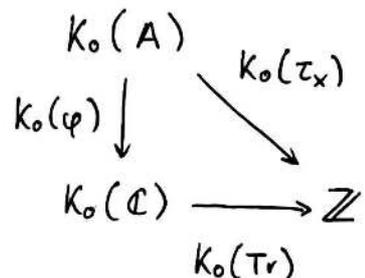
\* also true when  $H$  is not separable

ex let  $X$  be a cpct, connected Hdff space,  $A = C(X)$ . then for  $p \in P_n(A)$ ,  $p(x) \in M_n(\mathbb{C}) \forall x \in X$ . note that  $x \mapsto \text{Tr}(p(x))$  is in  $C(X, \mathbb{Z})$ , and since  $X$  is connected, this map is constant, i.e.  $\text{Tr}(p(x))$  is independent of  $x \in X$ . let  $\tau_x: C(X) \rightarrow \mathbb{C}$  be given by  $\tau_x(f) = f(x)$ . then  $\tau_x$  is a trace, hence we get a map  $K_0(\tau_x): K_0(A) \rightarrow \mathbb{C}$ . by uniqueness of trace,  $K_0(\tau_x)([p]_0) = \tau_x(p) = \text{Tr}(p(x))$ , hence  $K_0(\tau_x)$  is independent of  $x \in X$  and its image is in  $\mathbb{Z}$ . finally,  $1 = 1_A(x) = K_0(\tau_x)([1_A]_0)$ , hence  $K_0(\tau_x): K_0(A) \rightarrow \mathbb{Z}$  is surjective.

ex let  $X$  be as before + contractible, i.e.  $\exists x_0 \in X, \alpha: X \times I \rightarrow X$  s.t.  $\alpha(1, x) = x, \alpha(0, x) = x_0$ . for  $t \in [0, 1]$  define  $\varphi_t: C(X) \rightarrow C(X)$  given by  $\varphi_t(f)(x) = f(\alpha(t, x))$ . then  $\varphi_0(f)(x) = f(x_0)$  and  $\varphi_1 = \text{id}$ ; moreover,  $t \mapsto \varphi_t(f)$  is cts, hence  $\varphi_0 \sim_h \text{id}$ .

now let  $\varphi: C(X) \rightarrow \mathbb{C}, \psi: \mathbb{C} \rightarrow C(X)$  be given by  $\varphi(f) = f(x_0), \psi(\lambda) = \lambda \cdot 1$ .

then  $\varphi \circ \psi = \text{id}_{\mathbb{C}}$  and  $\psi \circ \varphi = \varphi_0 \sim_h \text{id}_A$ . by homotopy invariance,  $K_0(\varphi)$  is an isomorphism.



finally, the commutativity of the diagram on the right implies  $K_0(\tau_x)$  is a composition of isomorphisms, hence an isomorphism itself.