Correlators @ Finite Temperature

What is finite T?

$$\beta \equiv \frac{1}{T} = \frac{\partial S}{\partial E}$$

$$\boxed{\text{System}} \;\text{—}\; \boxed{\text{Bath}}$$

$$\mathcal{H}_S \ll \mathcal{H}_B$$

$\Rightarrow$ Don't know exact state of system

$\Rightarrow$ Average over thermal (canonical) ensemble of possible states

Thermal density matrix: $\rho \equiv e^{-\beta H}$

Hamiltonian of system, ignoring bath

Example: Electron in magnetic field

$$H = \frac{\omega}{2}\left(1 - \sigma_z\right) = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}$$

Ground state $|0\rangle = |\uparrow\rangle$

Excited state $|1\rangle = |\downarrow\rangle$

Heisenberg picture $\Rightarrow$ $z(t) \equiv e^{iHt}\sigma_z e^{-iHt} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$x(t) \equiv e^{iHt}\sigma_x e^{-iHt} = \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$$

$$\Rightarrow \langle z(t)\rangle \equiv \langle 0 | z(t) | 0 \rangle = 1$$

$$\langle x(t) \rangle = 0$$

Two-pt functions: $\langle z(t)\, z(0) \rangle = 1$

$$\langle x(t)\, x(0) \rangle = e^{-i\omega t}$$

Wightman function $\Rightarrow \langle x(0)\, x(t) \rangle = \langle x(t)\, x(0) \rangle^{\dagger} = e^{i\omega t}$

Can analytically continue to Euclidean time : $t \to -i\tau$

$$\Rightarrow \langle x(\tau)\, x(0) \rangle = e^{-\omega\tau}$$

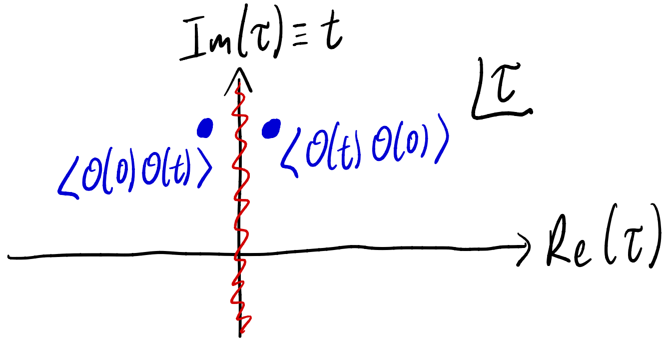
In QFT : energy spectrum unbounded

$$\Rightarrow \langle \mathcal{O}(\tau)\, \mathcal{O}(0) \rangle = \sum_n e^{-E_n \tau} |\langle \mathcal{O}(0) | n \rangle|^2 \to \infty \quad \text{for } \tau < 0$$

$\Rightarrow$ Euclidean correlators must be <u>time-ordered</u>

$$\langle T\{\mathcal{O}(\tau)\, \mathcal{O}(0)\} \rangle \equiv \langle \mathcal{O}(\tau)\, \mathcal{O}(0) \rangle\, \theta(\tau) + \langle \mathcal{O}(0)\, \mathcal{O}(\tau) \rangle\, \theta(-\tau)$$

More generally:



Finite temperature:

$$\Rightarrow \rho = e^{-\beta H} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta\omega} \end{pmatrix} = |0\rangle\langle 0| + e^{-\beta\omega} |1\rangle\langle 1|$$

$$\langle z(t)\rangle_\beta \equiv \frac{\mathrm{Tr}\left(e^{-\beta H} z(t)\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$

Partition function $Z(\beta)$

$$= \frac{\sum_n e^{-\beta E_n} \langle n| z(t) |n\rangle}{\sum_n e^{-\beta E_n}} = \frac{1 - e^{-\beta\omega}}{1 + e^{-\beta\omega}}$$

$\to 1$ @ $\beta \to \infty$

$\to 0$ @ $\beta \to 0$

$$\langle x(t)\rangle_\beta = 0$$

Thermal two-pt functions:

$$\langle z(t)\, z(0)\rangle_\beta = 1$$

$$\langle x(t)\, x(0)\rangle_\beta = \frac{e^{-i\omega t} + e^{-\beta\omega} e^{i\omega t}}{1 + e^{-\beta\omega}}$$

Euclidean correlator: $t \to -i\tau \Rightarrow \langle x(\tau)\, x(0)\rangle_\beta = \dfrac{e^{-\omega\tau} + e^{\omega(\tau-\beta)}}{1 + e^{-\beta\omega}}$

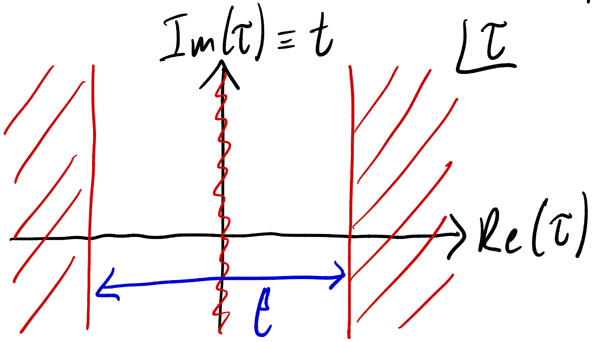
Shift $\tau \to \tau + \beta$:

$$\Rightarrow \langle x(\tau+\beta)\, x(0)\rangle = \frac{e^{-\omega(\tau+\beta)} + e^{\omega\tau}}{1 + e^{-\beta\omega}} = \langle x(0)\, x(\tau)\rangle_\beta$$

KMS condition

Alternative form: $\langle x(\tau)\, x(0)\rangle = \langle x(\beta-\tau)\, x(0)\rangle$

$\Rightarrow$ Thermal correlators live on a finite-width strip in complex plane



Time-ordered correlators are <u>periodic</u> in Euclidean (imaginary) time

<u>Example:</u> Harmonic oscillator

$$H = \omega a^\dagger a$$

$$x(t) \equiv e^{iHt}(a + a^\dagger)\, e^{-iHt}$$

$$\Rightarrow \langle x(t) \rangle = 0$$

$$\langle x(t)\, x(0) \rangle = e^{-i\omega t}$$

$$\rho = e^{-\beta H} = \sum_n e^{-n\beta\omega} |n\rangle\langle n|$$

$$\Rightarrow \langle x(t) \rangle_\beta \equiv \frac{\mathrm{Tr}\left(e^{-\beta H} x(t)\right)}{\mathrm{Tr}(e^{-\beta H})} = 0$$

$$\langle x(t)\; x(0) \rangle_\beta = \frac{e^{-i\omega t} + e^{-\beta\omega} e^{i\omega t}}{1 - e^{-\beta\omega}}$$

Bose statistics
(previous example = Fermi statistics)

(Infinite volume) QFT:

continuous spectrum

two equivalent representations

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_\beta = \int dE \, e^{-\beta E} \, {}_{in}\langle E | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | E \rangle_{in}$$

$$= \int dE \, e^{-\beta E} \, {}_{out}\langle E | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | E \rangle_{out}$$

$$|E\rangle_{in} \neq |E\rangle_{out} \Rightarrow S\text{-matrix} \equiv {}_{out}\langle E | E \rangle_{in}$$

Thermal correlators are inherently "in-in" (or "out-out") $\Rightarrow$ Schwinger-Keldysh

No ambiguity in finite volume: $|E\rangle = |E\rangle_{in} = |E\rangle_{out}$

What are in/out states?

$$H|\psi\rangle \equiv (H_0 + V)|\psi\rangle = E|\psi\rangle$$

free theory

interactions

$$\Rightarrow (E - H_0)|\psi\rangle = V|\psi\rangle$$

Multiply by $(E-H_0)^{-1}$ to obtain $|\psi\rangle$

"Lippmann-Schwinger" equation

$$\Rightarrow |\psi\rangle = |\phi\rangle + \frac{V}{E - H_0 \pm i\epsilon}|\psi\rangle$$

Eigenstate of H
$H|\psi\rangle = E|\psi\rangle$

Eigenstate of H_0
$H_0|\phi\rangle = E|\phi\rangle$

Ambiguity in $|\psi\rangle$ if H_0 has an eigenstate $|\phi\rangle$ w/ eigenvalue E

$\Rightarrow$ $\boxed{\text{in/out states}}$