Brauer Groups

Derived Functors

Let $F: \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor between two abelian categories.
i.e. given SES in $\mathcal{A}$
$$0 \to M \to N \to R \to 0$$
we have exact
$$0 \to F(M) \to F(N) \to F(R)$$

If $\mathcal{A}$ is "nice" we have a canonical way of extending this sequence, $\forall i \geq 1$ we have
$$R^i F: \mathcal{A} \to \mathcal{B}$$
the right derived functor, s.t.
$$0 \to F(M) \to F(N) \to F(R) \to R^1F(M) \to R^1F(N) \to R^1F(R)$$
$$\to R^2F(M) \to \dots$$
is exact.

<u>Ex</u>: Consider category of R-modules and functor $\mathrm{Hom}(A,-): R\text{-Mod} \to Ab$, the right derived functor is $\mathrm{Ext}^i_R(-,B)$

<u>Ex</u>: Group cohomology is the right derived functor of $(-)^G : k[G]\text{Mod} \to k[G]\text{mod}$.

<u>Def</u> For $q \in \mathbb{N}$, we define the functor
$$H^q(X, -) : \{\text{ab. sheaves on } X_\bullet\} \to Ab$$
as the qth derived functor of the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X)$. where $\mathcal{F}$ abelian sheaf

$H^q_\bullet(X, \mathcal{F})$ is called the $\bullet$ cohomology group of $\mathcal{F}$.

$$\bullet = \begin{cases} \text{Zariski} \\ \text{etale} \\ \text{fppf} \end{cases}$$

In particular given

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \quad \text{exact}$$

we have

$$0 \to H^0_\bullet(X, \mathcal{F}) \to H^0_\bullet(X, \mathcal{G}) \to H^0_\bullet(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to \dots$$

__Remark:__ Given $(X_i)_{i \in I}$ filtered inverse system of schemes w/ X_i qc qs and morphisms affine $X = \varprojlim X_i$ exists and where G_0 comm. group scheme, $G_i = G_0 \times_{X_0} X_i$, $G = G_0 \times_{X_0} X$

$$\varinjlim H^q_{et}(X_i, G_i) \cong H^q_{et}(X, G).$$

Cohomologies

~~Thm The functor~~

Def. Let $\mathcal{F}$ be a sheaf on $(\operatorname{Spec} k)_{et}$, we define

$$\mathcal{F}(k_s) = \varinjlim \mathcal{F}(L)$$

where our limit is taken over finite sep. ext. $L \supseteq k$ contained in k_s.

Thm. The functor

$$\{\text{sheaves of sets on } (\operatorname{Spec} k)_{et}\} \to \{G_k\text{-sets}\}$$
$$\mathcal{F} \longmapsto \mathcal{F}(k_s)$$

is an equivalence of categories.

Pf (sketch)

Given S a G_k set, for each $L \supseteq k$ finite sep. define

$$\mathcal{F}(L) = S^{Gal(k_s/L)}$$

We see this is our inverse.

Čech Cohomology

Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be an open covering on a site S. For $(i_0, \dots i_p) \in I^{p+1}$
$$U_{i_0 \dots i_p} = U_{i_0} \times_U U_{i_1} \times \dots \times_U U_{i_p}.$$
Forgetting jth factor gives us a projection
$$U_{i_0 \dots i_p} \to U_{i_0, \dots, \hat{i_j} \dots i_p}$$
Hence we obtain
$$\coprod_{i_0} U_{i_0} \leftleftarrows \coprod_{i_0 i_1} U_{i_0 i_1} \leftleftarrows \coprod_{i_0 i_1 i_2} U_{i_0 i_1 i_2} \leftleftarrows \cdots$$
Given $\mathcal{F}$ abelian presheaf on S we get
$$\prod_{i_0} \mathcal{F}(U_{i_0}) \rightrightarrows \prod_{i_0 i_1} \mathcal{F}(U_{i_0 i_1}) \rightrightarrows \cdots$$
We relabel these as C^0, C^1 etc. For each i we define $d^i : C^i \to C^{i+1}$ by taking the alternating sum of the arrows.

This gives us a complex.

The q^{th} Čech cohomology ~~is given by~~ of this complex is
$$\check{H}^q(\mathcal{U}, \mathcal{F}) = \frac{\ker d^q}{\operatorname{Im} d^{q-1}} \qquad q \geq 1$$
$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker d^0.$$

Def Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$, $\mathcal{V} = \{V_j \to U\}_{j \in J}$ be two open coverings on a site (C, T), $\mathcal{U}$ is called a refinement of $\mathcal{V}$ if $\exists \pi : I \to J$ and morphisms $U_i \to V_{\pi(i)}$ for each i.

Def. Given $\mathcal{F}$ an abelian presheaf on a site (C, T) $U \in C$ $q \in \mathbb{N}$ the qth Čech cohomology of U is
$$\check{H}^q(U, \mathcal{F}) = \varinjlim \check{H}^q(\mathcal{U}, \mathcal{F})$$

Why do we care?

Prop Given $\mathcal{F}$ an abelian sheaf on site $\mathcal{S}(C,T)$ $U \in \mathcal{C}$

$$\check{H}^0(U,\mathcal{F}) \cong H^0(U,\mathcal{F}) \cong \mathcal{F}(U)$$
$$\check{H}^1(U,\mathcal{F}) \cong H^1(U,\mathcal{F})$$
$$\check{H}^2(U,\mathcal{F}) \hookrightarrow H^2(U,\mathcal{F})$$

i.e. we have a way of computing low cohomologies.

Brauer Groups

We want to do cohomology of $\mathbb{G}_m$.

Def. $\mathbb{G}_m$ is representable by $\operatorname{Spec} \mathbb{Z}[t,t^{-1}]$ OR $\mathbb{G}_m(X) = \mathcal{O}_X(X)^\times$.

Prop Let X be scheme

i) $H^0_{zar}(X,\mathbb{G}_m) = H^0_{et}(X,\mathbb{G}_m) = H^0_{fppf}(X,\mathbb{G}_m) \cong \mathcal{O}_X(X)^\times$

ii) $H^1_{zar}(X,\mathbb{G}_m) \cong H^1_{et}(X,\mathbb{G}_m) \cong H^1_{fppf}(X,\mathbb{G}_m) = \operatorname{Pic} X$.

Cor (H90) Let $X = \operatorname{Spec} k$,

$$H^1(G_k, k_s^\times) = 0$$

Pf

Group cohomology on k is equiv to étale cohomology on $\operatorname{Spec} k$, $\operatorname{Pic} \operatorname{Spec} k = 0$.

Rmk For any smooth commutative abelian group scheme G over X

$$H^q_{et}(X,G) \cong H^q_{fppf}(X,G).$$

Def. Let X be a scheme, the Brauer group is

$$\operatorname{Br} X := H^2_{et}(X,\mathbb{G}_m).$$

Rmk By above $\operatorname{Br} X = H^2_{fppf}(X,\mathbb{G}_m)$

Rmk Given a morphism ~~of~~ of schemes $X \to Y$ we get an induced homo. $\mathrm{Br}\, Y \to \mathrm{Br}\, X$ hence a functor

$$\text{Schemes}^{op} \to \text{Ab}$$
$$X \to \mathrm{Br}\, X$$

Prop. Let X be a regular integral noetherian scheme
i) $\mathrm{Br}\, X \to \mathrm{Br}\, k(X)$ is injective
ii) $\mathrm{Br}\, X$ is torsion abelian.

Cor Let $X \to Y$ be a birational morphism of regular integral noetherian schemes, then $\mathrm{Br}\, Y \to \mathrm{Br}\, X$ is injective.

$$\begin{array}{ccc} \mathrm{Br}\, X & \xrightarrow{\text{inj}} & \mathrm{Br}\, k(X) \\ \uparrow & & \uparrow\wr \\ \mathrm{Br}\, Y & \xrightarrow{\text{inj.}} & \mathrm{Br}\, k(Y) \end{array}$$

since birational

Remark. Given $(X_i)_{i \in I}$ filtered system of inverse schemes X_i qcqs morphisms affine, $X = \varprojlim X_i$

$$\mathrm{Br}\, X \cong \varinjlim \mathrm{Br}\, X_i .$$

Remark When computing $\mathrm{Br}\, k$ for a field k, instead of cohomology we may define $\mathrm{Br}\, k$ as central simple k algebras ~~with $\otimes$ as our operation~~ up to some equivalence with operation $\otimes$.
Similarly we may define $\mathrm{Br}_{Az} X$ ~~as~~ - the Azumaya Brauer group - as equivalence classes of Azumaya algebras on X with operation tensor product (over $\mathcal{O}_X$)
We have $\mathrm{Br}_{Az} X \hookrightarrow \mathrm{Br}\, X$
and in some special cases (6.6.17)

$$\mathrm{Br}_{Az} X \cong (\mathrm{Br}\, X)_{tors}$$

If X regular quasi projective $\mathrm{Br}_{Az} X \cong \mathrm{Br}\, X$.

Computing

Spectral Sequences:

↳ Way of computing cohomology by computing cohomology of cohomology.

Ex For group cohomology we have the Lyndon-Hochschild ~~-Serre~~ spectral sequence ~~for~~ G profinite, $H \leq G$ normal closed this involves calculating

$$H^p(G/H, H^q(H, A))$$

for G-modules A.

↳ This gives rise to the inflation-restriction sequence

$$0 \to H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G,A) \xrightarrow{\text{Res}} H^1(H,A)^{G/H}$$
$$\to H^2(G/H, A^H) \to \ker(H^2(G,A) \to H^2(H,A))$$
$$\to H^1(G/H, H^1(H,A)) \to H^3(G/H, A^H).$$

We can do something similar for ~~schemes~~ varieties

Def Let X be a variety over a field k, let $X^s = X_{k^s}$, the algebraic part of the Brauer group of X is

$$Br_1(X) = \ker(Br X \to Br X^s).$$

~~S-S~~ LHS spectral sequence for varieties gives:

Prop Let X be a proper geometrically integral variety over k. Then the following is exact:

$$0 \to Pic X \to (Pic X^s)^{G_k} \to Br k \to Br_1 X \to H^1(G_k, Pic X^s)$$
$$\to H^3(k, \mathbb{G}_m)$$

Residue Homos.

Let R be a DVR w/ ~~fraction field K~~ field of fraction K. An element of $K^\times$ need not come from $R^\times$, this is measured by the valuation $K^\times \to \mathbb{Z}$.
Analogously an element of $\operatorname{Br} K$ need not come from $\operatorname{Br} R$, this is measured by a residue homo.

<u>Prop</u>. Let R be a DVR with field of fractions K and residue field k, we have an exact sequence
$$0 \to \operatorname{Br} R \to \operatorname{Br} K \xrightarrow{\text{res}} H^1(k, \mathbb{Q}/\mathbb{Z})$$
$\Leftrightarrow$ excluding <u>p-primary parts</u> from all groups if k imperfect of char p. elements of prime power order

<u>Construction</u> Replace R by its completion
After replacing $\operatorname{Br} K^{unr}$ by it w/o its p-primary part
$\operatorname{Br} K^{unr} = 0$
So $\operatorname{Br} K \cong H^2(\operatorname{Gal}(K^{unr}/K), (K^{unr})^\times)$ which maps to $H^2(\operatorname{Gal}(K^{unr}/K), \mathbb{Z})$ via valuation.
Also $\operatorname{Gal}(K^{unr}/K) \cong G_k$. So we apply LES to
$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

On a regular integral noetherian scheme X each integral divisor defines a discrete valuation v on $k(X)$. Taking all associated residue homo.s gives us a global version of the above.

<u>Thm</u> Let X be a regular integral noetherian scheme. Then
$$0 \to \operatorname{Br} X \to \operatorname{Br} k(X) \to \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbb{Q}/\mathbb{Z})$$
is exact (after removing p-primary parts)

where $X^{(1)}$ is codimension 1 points on X (in bijec with integral divisors).

Cor Let X, X' be nice birational varieties over k. Then $\mathrm{Br}\,X \cong \mathrm{Br}\,X'$ (~~up to prime~~ where we only consider prime to p parts where k imperfect of characteristic p).

Examples

• ~~If R valuation ring of non-arch local field, $\mathrm{Br}\,R = 0$~~

~~$\hookrightarrow$ follows from k finite so $\mathrm{Br}\,k = 0$~~

Prop Let R be complete local ring w/ residue field k. Then quotient map $R \to k$ induces isomorphism $\mathrm{Br}\,R \to \mathrm{Br}\,k$.

Cor If R valuation ring of non-arch local field $\mathrm{Br}\,R = 0$.

It follows that $\mathrm{Br}\,K \cong \mathbb{Q}/\mathbb{Z}$ where $K = FF(R)$. since $H^1(k, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$.

This is the inv map from class field theory.

Thm

• If X a proper curve over a ~~sep~~ separably closed field k, $\mathrm{Br}\,X = 0$

• Prop Let k be a field, $n \in \mathbb{Z}_{>0}$, then $\mathrm{Br}\,k \cong \mathrm{Br}\,\mathbb{P}^n_k$.

Pf Induction

Cor If X nice variety over k birational to $\mathbb{P}^n_k$ then $\mathrm{Br}\,k \cong \mathrm{Br}\,X$.