THERMAL CFTs, SYMMETRIES, BROKEN WARD IDENTITIES

CFT AT T=0

- we consider a quantum field theory invariant under conformal transformations (preserve angles):

  → translations: $P_\mu = i\partial_\mu$
  → rotations: $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
  → dilatation (scale invariance): $D = ix^\mu \partial_\mu$

  } affine transf. (maps lines to lines)

  → special conf. transf.: $K_\mu = i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$ ← the only non-affine (maps lines to circles and vice versa)

- we add the prescription for allowed transformations of the metric:

$$g_{\mu\nu} \longmapsto \Lambda(x) g_{\mu\nu} \qquad \text{Weyl transformations}$$

- above symmetries determine CFT at T=0; their generators (with appropriate commutation relations) form the conformal algebra
- interaction between fields is determined by correlation functions, which (because of conformal symmetry) need to obey Ward identities:

$$i \sum_i \delta(x_i - y) \langle \mathcal{O}_1(x_1) \ldots (\underset{\substack{\swarrow \\ \text{any generator}}}{G_a} \mathcal{O}_i(x_i)) \ldots \mathcal{O}_n(x_n) \rangle = \frac{\partial}{\partial y^\mu} \langle \underset{\substack{\swarrow \\ \text{Nöther current connected to that generator}}}{J^\mu_a(y)} \mathcal{O}_1(x_1) \ldots \mathcal{O}(x_n) \rangle$$

Example: (one-point function at T=0)

scale invariance: $\mathcal{O}(\lambda x) = \lambda^{\Delta_\mathcal{O}} \mathcal{O}(x)$

$\Delta_\mathcal{O}$: conformal dimension

$$\Rightarrow D\langle \mathcal{O} \rangle = (\underbrace{x^\mu \partial_\mu}_{\substack{=0 \\ \text{(by Ward identity} \\ \text{for translations)}}} + \Delta_\mathcal{O}) \langle \mathcal{O} \rangle \overset{\substack{\text{Ward identity} \\ \text{(for dilatation)}}}{=} 0$$

$$\Rightarrow \Delta_\mathcal{O} \langle \mathcal{O} \rangle = 0 \Rightarrow \Delta_\mathcal{O} = 0 \quad \text{or} \quad \langle \mathcal{O} \rangle = 0$$

$\Rightarrow$ all one-point correlation functions are 0, except the one of identity operator (with $\Delta_{\mathbb{1}} = 0$)

Higher point functions and the conformal data

- they can be expressed by lower-point functions via OPE:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle = \sum_{\mathcal{O}\in\mathcal{O}_1\times\mathcal{O}_2} c_{\mathcal{O}}(x_1)\langle\mathcal{O}(x_2)\mathcal{O}_3(x_3)\dots\mathcal{O}_n(x_n)\rangle$$

- What is therefore the necessary data we need to specify to characterise all correlation functions?

$$\{\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle\} \Leftrightarrow \{\{\underbrace{\Delta_{\mathcal{O}}}_{\text{conformal dim.}},\underbrace{J_{\mathcal{O}}}_{\text{spin}}\},\underbrace{f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}_{\text{structure constants (coeficients in OPE)}}\}$$ conformal data

THERMAL CFT

- the theory is no longer scale invariant (temperature $T=\frac{1}{\beta}$ determines the scale)
  - $\Rightarrow$ theory is less symmetric
- do we then need more constraints?
- no, it turns out correlation functions at finite temp. can be expressed in terms of corr. funct. at $T=0$:

$$\underbrace{\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\beta}_{\text{thermal corr. funct.}} = \frac{\mathrm{Tr}[e^{-\beta H}\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)]}{\mathrm{Tr}(e^{-\beta H})} \underset{\text{writting in terms of energy eigenvalues (\& eigenstates)}}{=} \frac{\sum_\alpha e^{-\beta E_\alpha}\overbrace{\langle E_\alpha|\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)|E_\alpha\rangle}^{\text{corr. funct. at } T=0}}{\sum_\alpha e^{-\beta E_\alpha}}$$

- it is hard to make use of this fact, because computation will require exact knowledge of infinite set of conformal data (this is beyond reach for any generic CFT)

Conformal data

- at $T=0$ two and three-point functions fully characterised the theory
- What about for finite temperature?
- again we can construct higher-point function using lower-point ones using OPE; but:

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_{\mathcal{O}} C_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}\mathcal{O} \quad \Rightarrow \langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = \sum_{\mathcal{O}} C_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}\langle\mathcal{O}\rangle$$

one-point functions are not non-zero only for identity!

we crucially used scale invariance for deriving this fact for $T=0$

- conformal data at $T>0$?

$$\text{thermal conformal data} \sim \{\{\Delta_{\mathcal{O}}, J_{\mathcal{O}}\}, f_{\mathcal{O}\mathcal{O}'\mathcal{O}''}, \langle\mathcal{O}^{\mu_1\ldots\mu_J}\rangle_\beta\}$$

difference from $T=0$

Convergence of OPE

- because of periodicity we have limited convergence radius (as opposed to $T=0$ theory)
- to see this, let's consider the following two-point function:

$$\langle\phi(\tau,\vec{x})\phi(0)\rangle_\beta \sim \frac{1}{(\tau^2+\vec{x}^2)^{2\Delta_\phi}}$$ (true for operators sufficiently close)

but using periodicity it follows:

$$\langle\phi(\beta-\tau,\vec{x})\phi(0)\rangle_\beta \sim \frac{1}{(\tau^2+\vec{x}^2)^{2\Delta_\phi}}$$

however, in last case no operator accounts for this divergence, so this is a sign of ill-definition, in this case being outside of convergence radius

$\Rightarrow$ so we can conclude that in case of above operators the convergence radius is

$$\tau^2+\vec{x}^2<\beta^2$$

it turns out to be true in general (non-trivial theorem)

Broken & unbroken Ward identities

- because of fixing the temperature some symmetries are broken, but can we write (broken) Ward identities that would preserve same amount of information?
- indeed we can derive them for any manifold $\mathcal{M}$ in exactly the same way as ordinary ones

$$i\sum_i \delta(x_i-y)\langle \mathcal{O}_1(x_1)\dots G_a\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}} = \nabla^y_\mu \langle J^\mu_a(y)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}}$$

broken Ward identities — only difference from normal Ward identities

- let's now choose $\mathcal{M} = S^1_\beta \times \mathbb{R}^{d-1}$ and integrate the above Ward identities:

$$i\sum_i \langle \mathcal{O}_1(x_1)\dots G_a\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle_\beta = \int_0^\beta dy_0 \int_{\mathbb{R}^{d-1}} d^{d-1}y\, \frac{\partial}{\partial y^\mu}\langle J^\mu_a(y)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\beta$$

$y_0=\tau$

in flat space:

$$=0$$

but for arbitrary manifold (including our thermal one) it is not necessary 0

- Since we integrated over $S^1_\beta\times\mathbb{R}^{d-1}$, we get two separate boundary terms:

(i) the one connected to $\int_{S^1_\beta}$ (boundary of): $\int_{\mathbb{R}^{d-1}} d^{d-1}y\langle[J^0_a(\beta,\vec{y})-J^0_a(0,\vec{y})]\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\beta$

(ii) the one connected to $\int_{\mathbb{R}^{d-1}}$:

$$\lim_{R\to\infty}\int_0^\beta d\tau\int_{S^{d-2}} d\Omega\, R^{d-2}\, n_i\langle J^i_a(\tau,R,\Omega)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\beta$$

- the second one produces IR divergences proportional to the volume
  $\Rightarrow$ we will just set it to 0

Clarification about periodicity:

- we have periodic coordinate τ: $\tau \equiv \tau+\beta$
- but it doesn't necessary mean that current at β is the same as current at 0
- why? Because we need to see it in connection to KMS condition (note that $\mathcal{O}_1$ and $\mathcal{O}_2$ are switched in 2.2.8)

for example: $\langle J(\beta,\vec{y})\mathcal{O}(x)\rangle \neq \langle J(0,\vec{y})\mathcal{O}(x)\rangle$

but rather:

$$=\langle \mathcal{O}(x)J(0,\vec{y})\rangle$$

So the difference is exactly the commutator

- in other words: holonomy is nonzero

  "if we go around a circle we acquire kind of 'charge'"

- So let's return to our Ward identities; let's denote $\Gamma_a^\beta(\vec{y}) = J_a^0(\beta,\vec{y}) - J_a^0(0,\vec{y})$ (which is non-zero according to discussion above)
- broken Ward identities:

$$i\sum_i \langle \mathcal{O}_1(x_1)\ldots G_a \mathcal{O}_i(x_i)\ldots \mathcal{O}_n(x_n)\rangle_\beta = \underbrace{\int_{\mathbb{R}^{d-1}} d^{d-1}y\, \langle \Gamma_a^\beta(y)\mathcal{O}_1\ldots\mathcal{O}_n\rangle_\beta}_{\text{breaking term}}$$

- Concrete transformations:
  - → translations (in both time and space): unbroken Ward identities, since the corresponding currents are periodic along the thermal circle ($\Gamma_{P_\mu}^\beta = 0$)
  - → spatial rotations: similarly, unbroken Ward identities; $\Gamma_{M_{ij}}^\beta = 0$
  - → dilatations: crucially broken as we introduce the scale

    breaking term: $\Gamma_D^\beta = \beta T^{00}(0,\vec{x})$
  - → boosts: broken as well; breaking term: $\Gamma_{M_{0i}}^\beta = \beta T^{0i}(0,\vec{x})$
  - → special conf. transf.: broken
- From these we can recognize interesting physical interpretation of breaking terms for dilatation and boosts:

★ $$H = \int_{\mathbb{R}^{d-1}} d^{d-1}x\,\left(T^{00}(\tau,\vec{x}) - \underbrace{\langle T^{00}\rangle_\beta}\right) \qquad \text{Hamiltonian}$$

(d-1)-dim Hamiltonian

by dilatation Ward id. for 1-pt. funct. this is fixed up to a single dimensionless coeff. b_T:

$$\langle T^{00}\rangle_\beta = -\frac{d-1}{d}\frac{b_T}{\beta^d}$$

⇒ broken Ward identity for dilatation therefore becomes:

$$H = -\frac{1}{\beta}D - E_0$$

comes from $\langle T^{00}\rangle_\beta$ term?

- the energy levels of this (d-1)-dim. Hamiltonian can be expressed with conformal spectrum of d-dim. CFT at T=0
- consider a basis $|i\rangle = \mathcal{O}_i|0\rangle$ (in d-dim. CFT, I suppose?)

  scaling operator
- matrix elements are:

  conf. dim. at T=0

$$\langle i|H|j\rangle_\beta = -\frac{\Delta_i}{\beta}\delta_{ij} - E_0$$

- energy levels: $E_n = -\frac{\Delta_n}{\beta} - E_0$ ← energy of thermal vacuum

★ $P^i = i \int_{\mathbb{R}^{d-1}} d^{d-1}x \, T^{0i}(\tau, \vec{x})$ spatial momentum operators

boost oper. M_{0i}

broken Ward id. for spatial rotations: $P_i = \frac{i}{\beta} L_i$

★ Ward id. for dilatations can be modified into:

$$\left(D + \underbrace{\beta \frac{\partial}{\partial \beta}}_{}\right) \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_\beta = 0$$

this term acts as Hamiltonian

★ implicit version of the Cardy formula

$$\frac{\partial}{\partial \beta} \langle \mathcal{O}(0) \rangle_\beta = -\frac{1}{\beta} \int d^{d-1}x \, \langle T^{00}(0, \vec{x}) \, \mathcal{O}(0) \rangle_\beta$$

changing β corresponds to inserting T^{00} in corr. funct.!

for $\mathcal{O} = T^{00}$: $\frac{\partial}{\partial \beta} \underbrace{\langle T^{00}(0) \rangle_\beta}_{} = -\frac{1}{\beta} \int d^{d-1}x \, \langle T^{00}(x) \, T^{00}(0) \rangle_\beta$

connected to
coeff. b_T, b_T is connected to
entropy

=> we implicitly get the formula for entropy