Talk 3 – Ander

Def $X \xrightarrow{f} Y$ is called flat at $x \in X$ if $\mathcal{O}_{\cancel{X},x}$ is flat as a $\mathcal{O}_{Y,f(x)}$-module

Def Faithfully flat = flat + surjective

Ex Any morphism $X \to \operatorname{Spec} k$ is flat

Prop Let $X \to S$ be a flat morphism of irreducible varieties. Then $\dim_x f = \dim X - \dim S$ $\forall x \in X$.

Def $X \xrightarrow{f} Y$ is called fppf if it is faithfully flat + locally finite presentation

$$\left(\frac{A[x_1, \dots, x_n]}{(f_1, \dots, f_m)} \longrightarrow A \right)$$

Prop $X \xrightarrow{f} Y$. If f is flat and of loc. finite presentation, f is open

Def $X \xrightarrow{f} Y$ is called fpqc if if is faithfully flat and if every qc open subset of Y is the image of a qc open of X

<u>Ex</u> Y. $X = \coprod_{y \in Y} \operatorname{Spec} \mathcal{O}_{Y,y}$

$$\left(\mathcal{O}_Y \longrightarrow \mathcal{O}_{Y,y} \Rightarrow \operatorname{Spec} \mathcal{O}_{Y,y} \to \operatorname{Spec} \mathcal{O}_Y \to Y \right)$$

$X \longrightarrow Y$

$$\coprod_{y \in Y} \underbrace{\operatorname{Spec} \mathcal{O}_{Y,y}}_{(0, m_y)} \xrightarrow{f} Y$$

$(0, m_y) \longmapsto y$

is faithfully flat, but not qc, because

~~for~~ any affine open $\operatorname{Spec} A \hookrightarrow Y$ (all affine are qc) the preimage is going to be

$$\left\{ \coprod_{y \in Y} \operatorname{Spec} \mathcal{O}_{Y,y} : y \in \operatorname{Spec} A \right\}$$

an infinite union of points (w/ discrete top.)

Def X is regular if it is locally noetherian and $\mathcal{O}_{X,x}$ is a regular local ring

(A local Noetherian ring R is called regular if $\dim R = \dim_{R/m} (m/m^2)$)

Rmk If X is of finite type over $\operatorname{Spec} \mathbb{Z}$, regularity is equivalent to the closed points of X regular.

Want X is smooth over $\mathbb{C}$

$$\Updownarrow$$

X^{an} is a manifold

Stupid idea "A $\mathbb{C}$-variety of dim r is smooth $\Longleftrightarrow$ if it is covered by Zariski opens, each isomorphic to an open subscheme $\mathbb{A}^r_{\mathbb{C}}$"

Why this fails $X = \operatorname{Spec} \frac{\mathbb{C}[x,y]}{(x^3+y^3-1)}$

$x^3 + y^3 = 1$ in $\mathbb{A}^2_{\mathbb{C}}$.

X^{an} is a manifold, but X can't be covered by opens iso. to $\mathbb{A}^1_{\mathbb{C}}$ because

$g(\mathbb{A}^1_{\mathbb{C}}) = 0$, $g(X) = 1$.

Rmk this will work with the étale "topology"

Def $X \xrightarrow{f} S$ is called smooth if:

- f is flat
- f is loc. of finite pres.
- $\forall s \in S$, $X_s = X \times_S \operatorname{Spec} k(s)$ is geom. regular over $k(s)$

Rmk $\operatorname{Spec} \dfrac{A[t_1, \dots, t_n]}{g(f_{r+1}, \dots, f_n)} \longrightarrow \operatorname{Spec} A$

is smooth (of rel. dimension r) iff

the matrix $\left(\dfrac{\partial g_i}{\partial t_j}(x) \right) \in M_{n-r \times n}(\kappa(x))$

has rank $n-r$.

Prop If $X \longrightarrow S$ of rel. dim. r

then $\Omega_{X/S}$ is locally free of

rank r. $\left(\Omega_{X/S}|_U \cong (\mathcal{O}_X|_U)^{\oplus r} \right)$

$\left(\operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k \Rightarrow \Omega_{k[x]/k} \cong (k[x]) \cdot dx \right)$

Non-ex Let k be a field of char

p, and let $X = \operatorname{Spec} k[\varepsilon]/(\varepsilon^p)$

$$\Omega_{X/k} = \frac{k[\varepsilon] \cdot \langle 1, d(\varepsilon^p) \rangle}{(\varepsilon^p)} \cong$$

$$\cong k[\varepsilon]/(\varepsilon^p) \Rightarrow \Omega_{X/k}$$

is locally free of rank 1, but

$\dim X - \dim \operatorname{Spec} k = 0 - 0 = 0$

<u>Prop</u> Let X be of locally finite type/k.

(i) X is smooth $\Longleftrightarrow$ X is geometrically regular

(ii) X is smooth $\Rightarrow$ regular, and if k is perfect: regular $\Rightarrow$ smooth

<u>Example</u> (regular variety that isn't smooth)

$k = \mathbb{F}_p(t)$, p odd, t a variable.

Let $X = \operatorname{Spec} \frac{k[x,y]}{(y^2 - (x^p - t))}$, $y^2 = x^p - t$

$f = y^2 - (x^p - t)$ is irreducible, and it's also irreducible in $\bar{k}[x,y] \Rightarrow X$ is geometrically integral

Let p be the point corresponding to the maximal ideal $(y, x^p - t)$. Then:

$$\{p\} = \left\{ (x,y) \in X : f(x,y) = 0,\ \frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0 \right\}$$

$\Rightarrow X$ is not smooth at $\{p\}$, but it's smooth everywhere else.

$$k(p) := \frac{\mathcal{O}_{X,p}}{m_p} \cong \frac{k[x,y]}{(x^p - t, y)}$$

$$\dim_{k(p)} m_p / m_p^2 \geq \dim_{k(p)} \frac{(x^p - t, y)}{(x^p - t, y)^2 + (f)}$$

$$\dim_{k(p)} \frac{(x^p - t, y)}{(x^p - t, y)^2} = 2 \quad \left(\text{generated by } x^p - t,\ y\right)$$

and the image of f in this v.s. is not zero:

$$\dim \frac{m_p}{m_p^2} = 1 = \dim X = \dim \mathcal{O}_{X,p}$$

$\Rightarrow$ X is regular at $\{p\}$

$$\operatorname{Spec} \frac{K[x,y]}{(x^p - t, y^2)} \underset{\operatorname{Spec} K}{\times} \operatorname{Spec} \overline{K} \simeq$$

$$\simeq \operatorname{Spec} \left(\frac{K[x,y]}{(x^p - t, y^2)} \otimes_K \overline{K} \right) \simeq$$

$$\simeq \operatorname{Spec} \frac{\overline{K}[x,y]}{(x^p - t, y^2)} \simeq \operatorname{Spec} \frac{\overline{K}[x,y]}{((x - t^{1/p})^p, y^2)}$$

$\Rightarrow$ nilpotents $\Rightarrow$ not reduced $\Rightarrow$ not regular (over $\overline{K}$)

Def $f: A \to B$ morphism of local rings is called unramified if $m_B = f(m_A)B$

(recall that we impose $f^{-1}(m_B) = m_A$ for local ring maps)

Non-

Ex $A = \mathbb{C}[[z]]$, $B = \mathbb{C}[[\sqrt{z}]]$

$A \overset{f}{\hookrightarrow} B$ inclusion. Then

$f(m_A)B = zB \neq \sqrt{z} \cdot B = m_B \Rightarrow$ not ur

Def $X \xrightarrow{f} S$ morphism of schemes is called ur at x if f is locally of finite type at x and $\mathcal{O}_{S,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is ur

Def f is G-ur if we replace "loc. of finite type" with "loc. of finite presentation"

Ex Open and closed immersions are ur, but not G - ur

<u>Example</u> Let L/K be some extension of number fields. A prime $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ is ramified $\iff$ $\operatorname{Spec} \mathcal{O}_L \xrightarrow{f} \operatorname{Spec} \mathcal{O}_K$ is ramified at $x = f^{-1}(\mathfrak{p})$

<u>Def</u> $X \xrightarrow{f} S$ is étale at $x \in X$ if if is flat and G - ur at x.

<u>Ex</u> $\operatorname{Spec} \coprod_{i=1}^{n} L_i \longrightarrow \operatorname{Spec} K$ is étale for L_i/K finite separable extension

<u>Non-ex</u> $X = \operatorname{Spec}(\overline{\mathbb{Q}}) \longrightarrow \operatorname{Spec}(\mathbb{Q})$ is flat, $\Omega_{X/\mathbb{Q}} = 0$ $\left(\frac{\mathbb{Q}[x]}{(f^2)} \right.$

$\left. d(f^2) = \underline{2f}\, df \right)$, but it's not loc. of finite presentation $\Rightarrow$ not étale

But it is "formally étale"

<u>Intuition</u> Étale morphisms are the analogue of locally biholomorphic maps in differential geometry.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

is étale $\iff$ $X^{an} \longrightarrow Y^{an}$ is locally biholomorphic

<u>Prop</u> Over a field k, and $X \xrightarrow{f} k$, TFAE:

- f is étale
- f is unramified
- $X = \coprod_{i \in I} \operatorname{Spec} L_i$, L_i/k is finite separable

<u>Corollary</u> If $X \xrightarrow{f} S$ is unramified, then $\dim_x f = \dim X - \dim S = 0$ $\forall x \in X$

Also, étale $\iff$ smooth of rel. dim $= 0$

Proof. Let $s = f(x)$. Being unramified is a property that's stable under base change

$$\Rightarrow X_s = X \times_S \operatorname{Spec} k(s) \text{ is ur over}$$

$\operatorname{Spec} k(s)$ $\Rightarrow \dim X_s = \dim\left(\bigsqcup_{i \in I} \operatorname{Spec} L_i\right) = 0$

Étale Cohomology

Motivation For a manifold X, singular cohom. groups $H^i(X, \mathbb{Z})$, $H^i(X, \mathbb{Z}/n\mathbb{Z})$ can be defined. Those behave well with analytic cohomology theories (de Rham, analytic sheaves ...)

Zariski topology: will work well for Coherent sheaves, but not for constant sheaves (because we have too few opens). For X "nice" curve of genus g:

$$H^1_{sing}(X(\mathbb{C}), \mathbb{Z}) \simeq H^1(X^{an}, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$$

but $H^1_{Zar}(X, \mathbb{Z}) = 0$ (because $\mathbb{Z}$ is a flasque sheaf on X)

For the étale cohomology, we still won't get good answers for $H^1_{ét}(X, \mathbb{Z})$, but we will get

$$H^1_{ét}(X, \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$$

for X some nice curve of genus g over an algebraically closed field s.t. $n \nmid \operatorname{char} K$. (this condition becomes redundant for fppf)

Also, we can define

$$H^1_{ét}(X, \mathbb{Z}_\ell) := \varprojlim_n H^1_{ét}(X, \mathbb{Z}/\ell^n)$$

(Assume small categories)

<u>Def</u> Let C be a category.

Consider all families of morphisms

$\{U_i \longrightarrow U\}_{i \in I}$ in C with a common target U. A Grothendieck topology on C is a set $\mathcal{T}$ whose elements are some of those families, called open coverings, satisfying:

(i) Isomorphisms are open coverings

(ii) An open covering of an open covering is an open covering

(iii) A base change of an open covering is an open covering.

<u>Def</u> A pair $(C, \mathcal{T})$, where C is a category and $\mathcal{T}$ is a Grothendieck topology on C is called a <u>site</u>

Ex Let X be a top. space,

$\mathcal{C} = \mathrm{Ouv}(X)$, and

$$\mathrm{Hom}_{\mathcal{C}}(U,V) = \begin{cases} \{i\}, & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

$\overline{\mathcal{C}}$ is the collection of families

$\{U_i \to U\}_{i\in I}$ s.t. $\bigcup_{i\in I} U_i = U$

the (small) Zariski site X_{Zar}

The (small) étale site Let X be a scheme. We define $\mathcal{C} = \mathrm{Et}_X$, where objects are schemes U together with an étale morphism $U \to X$, and morphisms are X-morphisms

$$\left(\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[dl] \\ X & \end{tikzcd}\right)$$

τ is the collection of families $\{\phi_i : U_i \to U\}$ of morphisms in C s.t. $\bigcup_{i\in I} \phi_i(U_i) = U$ as top. spaces

Similarly, one can define the (big) fpqc and fppf sites, by imposing that $\coprod_{i\in I} U_i \longrightarrow U$ is fppf/fpqc

X_{fppf}, X_{fpqc}

<u>Notation</u> By étale site, we always mean the small étale site.

<u>Rmk</u> We have (cont.) maps

$$X_{fpqc} \to X_{fppf} \to X_{et} \to X_{Zar}$$

<u>Cohomology</u> We use derived functors

($\bullet \in \{Zar, et, fppf\}$)

<u>Def</u> For $q \in \mathbb{N}$, define the functor

$$\{\text{abelian sheaves on } X_\bullet\} \longrightarrow Ab$$

$$F \longmapsto H^q_\bullet(X, F)$$

as the q-th right derived functor of the left-exact global sections functor

$$F \longmapsto F(X),$$ and we call

$H^q_\bullet(X, F)$ the q-th Zariski / étale / fppf

cohomology group of F