Deformation of Rings

Goals: 1) Def. of $X = \operatorname{Spec} B$ given $T^1(B/k, B)$

2) Def of nonsing. X given $H^1(X, \mathcal{T}_X)$

tangent sh.

<u>Def</u>. let X be a scheme over k,

let A - Artin Ring $/k$.

Deformation of X over A is a pair of

a scheme X', flat over, $i : X \hookrightarrow X'$:

$$i \times_A k : X \longrightarrow X' \times_A k \quad \text{- isomorphism.}$$

<u>Def</u>. X_1, i_1 and X_2, i_2 - 2 def. of X.

$(X_1, i_1) \sim (X_2, i_2)$ if $\exists$ iso $f : X_1 \longrightarrow X_2$:

$$\begin{tikzcd} X_1 \arrow[rr, "f"] & & X_2 \\ & X \arrow[ul, hook, "i_1"] \arrow[ur, hook, "i_2"'] & \end{tikzcd}$$

commutes.

Affine schemes

Let B - k-algebra

Def. of $\operatorname{Spec} B$ over $D = k[x]/(x^2)$ - D-algebra

B' - flat over D with homom-m $B' \twoheadrightarrow B$:

$$B' \otimes_D k \longrightarrow B$$

B' - flat $\Leftrightarrow$ $0 \longrightarrow \underbrace{B}_{\text{ideal: } B^2 = 0} \xrightarrow{i} \underbrace{B' \longrightarrow B}_{} \longrightarrow 0$

$(2,2)$

$$0 \to (x) \to D \to k \to 0$$

$B \simeq B' \otimes k$ flat

$$B \otimes (x) \hookrightarrow B'$$

<u>More general</u>:

A - ring, B - A-algebra, M - B-module

<u>Def</u>. <u>Extansion</u> of B by M as A-algebras

is exact sequence:

$$0 \longrightarrow M \longrightarrow B' \longrightarrow B \longrightarrow 0$$

<u>Def</u>. let B' and B'' - be 2 extansion of B

b M, we say $B' \sim B''$ if $\exists f : B' \to B''$ - iso :

$$\begin{array}{ccccccccc} 0 & \to & M & \to & B' & \to & B & \to & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\ 0 & \to & M & \to & B'' & \to & B & \to & 0 \end{array}$$

triv extantion $\quad B' = M \oplus B$

Theorem:

Equiv. classes of extansions of B by M as A-algebras in 1-to-1 correspondance with elements of $T^1(B/A, M)$.

Proof: $A[x_1, \dots, x_n] \to B$

We have $A[x] \to B$ - surj.

$y = \{y_i\}$ - with the same indeces as generators of M.

B' - extansion of B $\Rightarrow$ surj $f : A[x, y] \to B'$

$$\begin{tikzcd}
0 \arrow[r] & (y) \arrow[r] \arrow[d] & A[x,y] \arrow[r] \arrow[d, "f"'] & A[x] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Plan

1.) Classify quotients of $A[x, y]$

2) for a given B' how many dif. ways two present it as a quotient of $A[x, y]$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (y) \arrow[r] \arrow[d] & A[x,y] \arrow[r] \arrow[d] & A[x] \arrow[r] \arrow[d] \arrow[l, bend left] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

splits

$\Updownarrow$

def. I' $\Leftrightarrow$ def. $\varphi \in \mathrm{Hom}(I, M) = \mathrm{Hom}(I/I^2, M)$

Now using thm 4.5 consider

$f : A[x,y] \to B'$ def. $\bar{f} : A[x] \to B'$

$\{\bar{f}\}$ homogeneous under action of

$\mathrm{Der}_A(A[x], M)$

$\Rightarrow \{f\}$ - also homogeneous.

$$A \to A[x] \to B$$

$$T^0(A[x]/A, M) = \mathrm{Der}_A(A[x], M) \to \mathrm{Hom}$$
$$\to T^1(B/A[x], M) \to$$
$$\to T^1(B/A[x], M) \to$$
$$\to T^1(B/A, M) \to$$
$$\to T^1(A[x]/A, M) = 0$$

3.5 $A \to B \to C$

$$T^0(B/A, M) \to$$
$$\to T^1(C/B, M) \to$$
$$\to T^1(C/A, M) \to T^1(B/A, M)$$

3.6

$$T^0(B/A, M) = \mathrm{Der}_A(B, M)$$

3.7 $B = A[x]$

$$T^i(B/A, M) = 0, \quad i = 1, 2$$

3.8 $A \twoheadrightarrow B$, I - ker

$$T^1(B/A, M) = \mathrm{Hom}(I/I^2, M)$$

$$\mathrm{Der}_A(A[x], M) \xrightarrow{f} \mathrm{Hom}(I/I^2, M) \to T^1(B/A, M) \to 0$$

$$T^1(B/A, M) = \mathrm{Hom}(I/I^2, M)/\mathrm{Im} f$$

__Corollary__: k - field, B - k-algebra

$$\{\text{Def of } B \text{ over } D\} \longleftrightarrow \{\text{el. of } T^1(B/k, M)\}$$

Nonsing schemes

Theorem

let X - nonsing shceme $/k$.

$$\left\{ \begin{array}{l} \text{Def of } X \\ \text{over } D \end{array} \right\} \longleftrightarrow \left\{ \text{el. of } H^1(X, \mathcal{T}_X) \right\}$$

tang. sheaf

Proof:

let X' - be def. of X, $\mathcal{U} = (\mathcal{U}_i)$ - open affine covering

Over every $\mathcal{U}_i$ def. $\mathcal{U}_i'$ is trivial $\Rightarrow$

$$\varphi_i : \mathcal{U}_i \times_k D \xrightarrow{\sim} \mathcal{U}_i'$$

Consider $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$

$\psi_{ij} = \varphi_j^{-1} \circ \varphi_i$ - automorphism of $\mathcal{U}_{ij} \times_k D$

$$\rightsquigarrow \Theta_{ij} \in H^0(\mathcal{U}_{ij}, \mathcal{T}_X)$$

On $\mathcal{U}_{ijk}$ we get $\Theta_{ij} + \Theta_{jk} + \Theta_{ki} = 0$

$\psi_{ki} \circ \psi_{jk} \circ \psi_{ij} = id$

$\Rightarrow \Theta_{ij}$ - 1 - Čech cocycle

$\psi_{ij} = \underbrace{\varphi_j^{-1}} \circ \varphi_i$ if we replace φ_j with φ_j'

we get $\psi'_{ij} = \varphi_j'^{-1} \circ \varphi_i$

$\rightsquigarrow \theta'_{ij} = \theta_{ij} - (\alpha_j - \alpha_i)$, $\alpha_j, \alpha_i \in H^0(\mathcal{U}_{ij}, T_X)$

$\Rightarrow \theta_{ij} \in \check{H}^1(\mathcal{U}_{ij}, T_X)$

and in our case it is the same as normal H^1.

Example:

1) $X = \mathbb{P}^n$, $n \geq 1 \Rightarrow H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$

2) Curve C, $g(C) \geq 2$

$H^1(T_C)$ dual $H^0(\Omega_C^{\otimes 2})$

$\ell(D) - \underbrace{\ell(k-D)}_{\overset{\shortparallel}{0}} = \underset{\overset{\shortparallel}{2(2g-2)}}{\deg D} - g + 1$

$D = 2k$

$\ell(2k) = 4g - 4 - g + 1 = 3g - 3$