Chapter 8 - The K_1-group

Notation: $\mathcal{U}_n(A) := \mathcal{U}(M_n(A))$

(A unital) $\mathcal{U}_\infty(A) := \bigcup_{n\geq 1} \mathcal{U}_n(A)$

Def $u \in \mathcal{U}_n(A)$, $v \in \mathcal{U}_m(A)$.

$u \sim_1 v$ if $\exists\ k \geq \max\{n,m\}$ s.t. $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$.

$K_1(A) := \mathcal{U}_\infty(\tilde{A})/_{\sim_1}$ with $[u]_1 + [v]_1 := [u \oplus v]_1$.

Prop. $K_1(A)$ is an abelian group with unity $[1_{\tilde{A}}]_1 = 0$.

Moreover, if $u, v \in \mathcal{U}_n(\tilde{A})$, then

(i) $u \sim_h v \Rightarrow [u]_1 = [v]_1$.

(ii) $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$ $\rightsquigarrow$ basically Whitehead Lemma.

Rmk: Using polar decomposition, we can use $GL_\infty(\tilde{A})$ instead of $\mathcal{U}_\infty(\tilde{A})$:

$K_1(A) \cong GL_\infty(\tilde{A})/_{\sim_1}$.

Prop (Universal prop. of K_1). G ab. grp., $\nu : \mathcal{U}_\infty(\tilde{A}) \to G$ s.t.

(i) $\nu(u \oplus v) = \nu(u) + \nu(v)$;

(ii) $\nu(1_{\tilde{A}}) = 0$;

(iii) $u, v \in \mathcal{U}_n(\tilde{A})$, $u \sim_h v \Rightarrow \nu(u) = \nu(v)$.

Then $\exists!\ \alpha : K_1(A) \to G$ s.t.

$$\begin{tikzcd} \mathcal{U}_\infty(\tilde{A}) \arrow[d, "{[\cdot]_1}"'] \arrow[dr, "\nu"] & \\ K_1(A) \arrow[r, dashed, "\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & G. \end{tikzcd}$$

__Sketch of Proof__: Define $\alpha([u]_1) := \nu(u)$.
↳ choose a representative.

Note $\nu(1_r) = \nu(1 \oplus \cdots \oplus 1) = 0$. ($\forall r \in \mathbb{N}$)

__Claim__: α is well-defined.

$\underset{\mathcal{U}_n(\tilde{A})}{\underset{\cap}{[u]_1}} = \underset{\mathcal{U}_m(\tilde{A})}{\underset{\cap}{[v]_1}} \Rightarrow \exists\, k \geq \max\{n, m\}$ s.t. $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m} \Rightarrow$

$\Rightarrow \nu(u) = \nu(u \oplus 1_{k-n}) = \nu(v \oplus 1_{k-m}) = \nu(v)$ //

Moreover, the diagram commute, α is a homomorphism and is unique (use the fact that $[-]_1$ is surjective). ■

Prop A unital $\Rightarrow K_1(A) \cong \mathcal{U}_\infty(A)/_{\sim_1}$. [short version].

long version of the same theorem...

Prop A unital $\Rightarrow \exists\, \rho : K_1(A) \to \mathcal{U}_\infty(A)/_{\sim_1}$ iso. s.t.

$$\begin{tikzcd} \mathcal{U}_\infty(\tilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{U}_\infty(A) \arrow[d, "{[\cdot]_1}"] \\ K_1(A) \arrow[r, "\rho"'] & \mathcal{U}_\infty(A)/_{\sim_1} \end{tikzcd}$$

, where μ is obtained extending

$\mu : \tilde{A} \to A$
$a + \alpha(1_{\tilde{A}} - 1_A) \mapsto a$

to $\mathcal{U}_\infty(A)$.

Coro: $K_1(A) \cong K_1(\tilde{A})$ for every C^*-alg. A.

Ex: $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = 0$, $K_1(B(\mathcal{H})) = 0$ $\forall$ Hilb. sp $\mathcal{H}$.

Since $\mathcal{U}(M_n(\mathbb{C}))$ is connected $\forall n \geq 1$ (Coro 2.1.4)

More generally, $U \sim_n 1$ $\forall U \in \mathcal{U}_n(B(\mathcal{H}))$.

Indeed, $\varphi : \mathbb{T} \to [0, 2\pi)$, $e^{i\theta} \mapsto \theta$ is Borel and $\left[z = e^{i\varphi(z)} \ \forall z \in \mathbb{T} \right]$ $\overset{\text{Functional calculus}}{\rightsquigarrow}$ $U = e^{i\varphi(U)}$ with $\varphi(U)$ self adjoint.

$\mathcal{U}_n(B(\mathcal{H})) \cong \mathcal{U}(B(\mathcal{H}^n)) \xrightarrow{\text{Prop. 2.1.6.}} U \sim_n 1.$

§8.2 - Functoriality of K_1

$\left[\varphi \rightsquigarrow \tilde{\varphi} : \tilde{A} \to \tilde{B} \rightsquigarrow \tilde{\varphi}_n : M_n(\tilde{A}) \to M_n(\tilde{B}) \ \forall n \right]$

$\underset{*\text{-hom}}{\varphi : A \to B} \overset{\text{extends to}}{\rightsquigarrow} \tilde{\varphi} : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(\tilde{B})$.

Define $\nu : \mathcal{U}_\infty(\tilde{A}) \to K_1(B)$, $U \mapsto [\tilde{\varphi}(U)]_1$ $\overset{\text{universal prop.}}{\rightsquigarrow}$ $\exists\, K_1(\varphi) : K_1(A) \to K_1(B)$.

Prop (K_1 is a functor).

(i) $K_1(id_A) = id_{K_1(A)}$.

(ii) $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$; $\varphi : A \to B$, $\psi : B \to C$.

(iii) $K_1(0) = 0$. (iv) $K_1(0_{B,A}) = 0_{K_1(B), K_1(A)}$.

(v) $\varphi, \psi : A \to B$, $\varphi \sim_h \psi \implies K_1(\varphi) = K_1(\psi)$.

(vi) A, B homotopic equiv. $\implies K_1(A) \cong K_1(B)$.

"Proof" : (ii) Follows from $\widetilde{\psi \circ \varphi} \underset{\tilde{A}}{=} \tilde{\psi} \circ \tilde{\varphi}$.

(iii) $K_1(\{0\}) \cong K_1(\underbrace{\widetilde{\{0\}}}_{\cong \mathbb{C}}) = 0$.

<u>Lemma</u>. $\varphi : A \to B$ surj. $*$-hom, $g \in \ker K_1(\varphi)$.

$\Longrightarrow \exists\, u \in \mathcal{U}_\infty(\tilde{A})$ s.t. $g = [u]_1$ and $\tilde{\varphi}(u) = 1$.

<u>Prop</u> (K_1 is half-exact) $0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$ s.e.s.

$\Longrightarrow K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B)$ is exact.

<u>Proof</u>. $\psi \circ \varphi = 0 \Longrightarrow K_1(\psi) \circ K_1(\varphi) = 0 \Longrightarrow \operatorname{Im} K_1(\varphi) \subseteq \operatorname{Ker} K_1(\psi)$.

Also,

$g \in \operatorname{Ker} K_1(\psi) \Longrightarrow \exists\, u \in \mathcal{U}_n(\tilde{A})$ s.t. $g = [u]_1$ and $\tilde{\psi}(u) = 1$.

$\xrightarrow{\text{Lemma 4.3.1}} \exists\, v \in \mathcal{U}_n(\tilde{I})$ s.t. $\tilde{\varphi}(v) = u$

$\therefore [v]_1 \in K_1(I)$ and $K_1(\varphi)([v]_1) = [\tilde{\varphi}(v)]_1 = [u]_1 = g$.

$\Longrightarrow \operatorname{Ker} K_1(\psi) \subseteq \operatorname{Im} K_1(\varphi)$.

→ maybe just a remark!

<u>Prop</u> (K_1 is split-exact) $0 \longrightarrow I \xrightarrow{\varphi} A \underset{\lambda}{\overset{\psi}{\rightleftarrows}} B \longrightarrow 0$ split exact

$\Longrightarrow 0 \longrightarrow K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \underset{K_1(\lambda)}{\overset{K_1(\psi)}{\rightleftarrows}} K_1(B) \longrightarrow 0$ split exact.

<u>Prop</u> (K_1 preserve direct sums) $K_1(A \oplus B) \cong K_1(A) \oplus K_1(B)$.

Extras

Digression - Inductive Limits (in Ab and $C^*\text{-alg}$).

$\dashrightarrow$ $(*\text{-})$hom

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[drr, "\mu_1"'] & A_2 \arrow[r, "\varphi_2"] \arrow[dr, "\mu_2"] & A_3 \arrow[r] \arrow[d, "\mu_3"] & \cdots \\
 & & A &
\end{tikzcd}$$

] inductive seq.

] $(A, \{\mu_n\}) =:$ system inductive limit

Universal prop.

$$\begin{tikzcd}
A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\mu_n"] \arrow[ddr, bend right, "\lambda_n"'] & & A_{n+1} \arrow[dl, "\mu_{n+1}"'] \arrow[ddl, bend left, "\lambda_{n+1}"] \\
 & A \arrow[d, dashed, "\lambda"] & \\
 & B &
\end{tikzcd}$$

$\forall (B, \{\lambda_n\})$ system

$\exists! \ \lambda : A \to B \ \ldots$

Notation: $A = \varinjlim A_n$.

Idea of construction Denote $\varphi_{mn} = \varphi_{m-1} \circ \cdots \circ \varphi_n \quad (m > n)$

$\varphi_{nn} = id_{A_n}$

$$A := \{ (a_n) \in \prod_{i \geq 1} A_i \ / \ \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0 \Rightarrow a_n = \varphi_{nn_0}(a_{n_0}) \}$$ "predictable tails"

with

$$\mu_m : A_m \longrightarrow A$$
$$a \longmapsto (a_n) \text{ with } a_n = \begin{cases} 0 & , n < m; \\ \varphi_{nm}(a), & n \geq m. \end{cases}$$

In $C^*\text{-Alg}$, $\alpha((a_n)) := \limsup_{n \geq n_0} \|\varphi_{nn_0}(a_{n_0})\|$: C^*-seminorm.

Define $\varinjlim A_n := \overline{A / N_\alpha}$, where $N_\alpha = \{ (a_n) \in A : \alpha((a_n)) = 0 \}$

Ex ① $A \xrightarrow{\iota_1} M_2(A) \xrightarrow{\iota_2} M_3(A) \longrightarrow \cdots$ $\qquad \varinjlim M_n(A) \cong A \otimes \mathbb{K}$

$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

compact op. of a separable Hilb. sp.

② $\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \cdots$ $\qquad \varinjlim \mathbb{Z} \cong \mathbb{Q}$.

Def An AF-algebra is a C^*-alg that is the inductive limit of finite dimensional C^*-alg.

$i \in \{0,1\}$

Prop (K_i is continuous) $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \longrightarrow \cdots$ ind. seq.

$\Longrightarrow \quad K_i(\varinjlim A_n) = \varinjlim K_i(A_n)$.

Prop $K_i(A) \cong K_i(M_n(A))$ and $K_i(A) \cong K_i(A \otimes \mathbb{K})$ with isomorphisms induced by $\lambda_n : A \to M_n(A)$ and $\varphi : A \longrightarrow A \otimes \mathbb{K}$.

$a \mapsto \mathrm{diag}(a, 0, \dots, 0) \qquad a \longmapsto (\iota_{n1}(a))$

Ex $K_1(\mathbb{K}) \cong K_1(\mathbb{C}) = 0$.

$\cong \mathbb{K} \otimes \mathbb{C}$

Cor $A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Longrightarrow K_i(A) \cong K_i(B)$.

(stably isomorphic)

Thm (Elliott) An unital AF-alg A is "completely determined" by $(K_0(A), K_0(A)^+, [1_A]_0)$.

$\hookrightarrow \{[p]_0 : p \in \mathcal{P}_\infty(A)\}$ "positive cone".

Remark (§8.3) $A = C(X)$, X cpt. HFF.

$\Longrightarrow K_1(C(X)) \cong \pi^1(X) \oplus \mathrm{Ker}(\Delta)$ for a "nice" map $\Delta : K_1(C(X)) \longrightarrow \pi^1(X)$,

defined using a "determinant".

where $\pi^1(X) = \mathcal{U}(C(X)) / \mathcal{U}_0(C(X))$

$\hookrightarrow$ cohomotopy

Example: $\pi^1(\mathbb{T}) \cong \mathbb{Z}$

In particular, $K_1(C(\mathbb{T})) \neq 0$.