

Chapter 8 - The K_1 -group

(1)

Notation: $U_n(A) := U(M_n(A))$
(A unital) $U_\infty(A) := \bigcup_{n \geq 1} U_n(A)$

Def $U \in U_n(A), V \in U_m(A)$.

$U \sim_n V$ if $\exists K \geq \max\{n, m\}$ st. $U \oplus 1_{K-n} \sim_n V \oplus 1_{K-m}$

$K_1(A) := U_\infty(\tilde{A}) / \sim_1$ with $[U]_1 + [V]_1 := [U \oplus V]_1$.

Prop. $K_1(A)$ is an abelian group with unity $[1_A]_1 = 0$.

Moreover, if $U, V \in U_n(\tilde{A})$, then

(i) $U \sim_n V \Rightarrow [U]_1 = [V]_1$.

(ii) $[UV]_1 = [VU]_1 = [U]_1 + [V]_1$ \rightsquigarrow basically Whitehead Lemma.

Remark: Using polar decomposition, we can use $GL_\infty(\tilde{A})$

instead of $U_\infty(\tilde{A})$;

$K_1(A) \cong GL_\infty(\tilde{A}) / \sim_1$

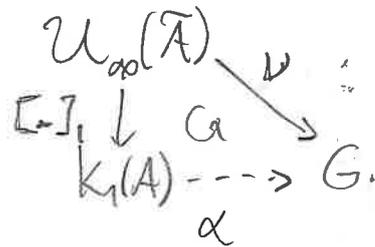
Prop (Universal prop. of K_1). G ab. grp., $V: U_\infty(\tilde{A}) \rightarrow G$ s.t.

(i) $V(U \oplus V) = V(U) + V(V)$;

(ii) $V(1) = 0$;

(iii) $U, V \in U_n(\tilde{A}), U \sim_n V \implies V(U) = V(V)$.

Then $\exists! \alpha: K_1(A) \rightarrow G$ s.t.



Sketch of Proof: Define $\alpha([U]_1) = V(U)$.

\hookrightarrow choose a representative.

Note $V(1_r) = V(1 \oplus \dots \oplus 1) = 0$. ($\forall r \in \mathbb{N}$)

Claim: α is well-defined.

$[U]_1 = [V]_1 \implies \exists k \geq \max\{n, m\}$ s.t. $U \oplus 1_{k-n} \sim_n V \oplus 1_{k-m} \implies$

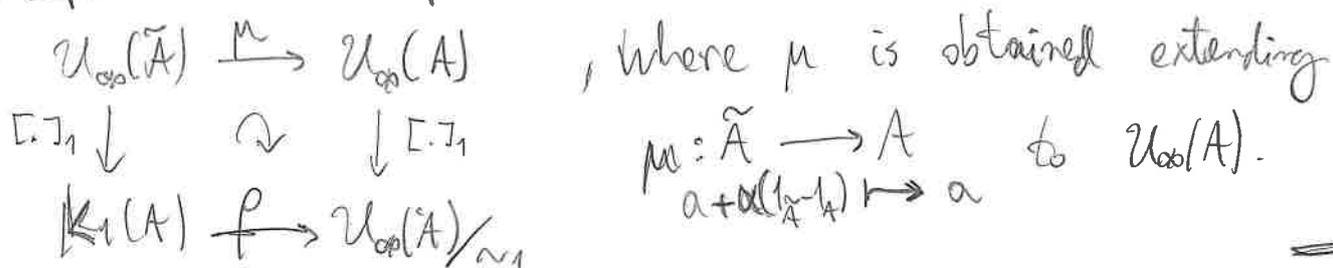
$\implies V(U) = V(U \oplus 1_{k-n}) = V(V \oplus 1_{k-m}) = V(V) //$

Moreover, the diagram commutes, α is a homomorphism and is unique (use the fact that $[\cdot]_1$ is surjective). \blacksquare

Prop. A unital $\implies K_1(A) \cong U_{\text{od}}(A) / \sim_1$. [short version].

[long version of the same theorem...]

Prop A unital $\implies \exists \rho: K_1(A) \rightarrow U_{\text{od}}(A) / \sim_1$ iso. s.t.



Coro: $K_1(A) \cong K_1(\tilde{A})$ for every C^* -alg. A .

Ex: $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = 0$, $K_1(B(\mathcal{H})) = 0 \quad \forall$ Hilb. sp \mathcal{H} .

Since $U(M_n(\mathbb{C}))$ is connected $\forall n \geq 1$ (Coro 2.1.4)

More generally, $U \sim_n 1 \quad \forall U \in U_n(B(\mathcal{H}))$.

Indeed, $\varphi: \mathbb{T} \rightarrow [0, 2\pi)$ is Borel and Functional calculus $U = e^{i\varphi(U)}$ with $\varphi(U)$ self adjoint.
 $e^{i\theta} \mapsto \theta$
 $\begin{cases} z = e^{i\varphi(z)} \\ \forall z \in \mathbb{T} \end{cases}$

$U_n(B(\mathcal{H})) \cong U(B(\mathbb{C}^n))$
 $\xrightarrow{\text{Prop. 2.1.6.}} U \sim_n 1.$

§8.2 - Functoriality of K_1

$\varphi \rightsquigarrow \tilde{\varphi}: \tilde{A} \rightarrow \tilde{B}$
 $\rightsquigarrow \tilde{\varphi}_n: M_n(\tilde{A}) \rightarrow M_n(\tilde{B}) \quad \forall n$

$\varphi: A \rightarrow B$ \rightsquigarrow $\tilde{\varphi}: U_\infty(\tilde{A}) \rightarrow U_\infty(\tilde{B})$.
 \ast -hom extends to

Define $\nu: U_\infty(\tilde{A}) \rightarrow K_1(B)$ \rightsquigarrow $\exists K_1(\varphi): K_1(A) \rightarrow K_1(B)$.
 $\nu \mapsto [\tilde{\varphi}(\nu)]_1$ universal prop.

Prop (K_1 is a functor).

- (i) $K_1(id_A) = id_{K_1(A)}$.
- (ii) $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$; $\varphi: A \rightarrow B, \psi: B \rightarrow C$.
- (iii) $K_1(0) = 0$. (iv) $K_1(0_{B,A}) = 0_{K_1(B), K_1(A)}$.
- (v) $\varphi, \psi: A \rightarrow B, \varphi \sim_n \psi \implies K_1(\varphi) = K_1(\psi)$.
- (vi) A, B homotopic equiv. $\implies K_1(A) \cong K_1(B)$.

"Proof" (2) Follows from $\tilde{\varphi} \circ \varphi = \tilde{\varphi} \circ \varphi$.

(9.1) $K_1(\text{Im } \varphi) \cong K_1(\text{Im } \tilde{\varphi}) = 0$

Lemma. $\varphi: A \rightarrow B$ surj. \ast -hom., $g \in \ker K_1(\varphi)$.
 $\Rightarrow \exists v \in U_{\infty}(A)$ s.t. $g = [v]$, and $\tilde{\varphi}(v) = 1$.

Prop (K_1 is half-exact) $0 \rightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \rightarrow 0$ s.e.s.
 $\Rightarrow K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B)$ is exact.

Proof. $\varphi \circ \psi = 0 \Rightarrow K_1(\psi) \circ K_1(\varphi) = 0 \Rightarrow \text{Im } K_1(\varphi) \subseteq \ker K_1(\psi)$.
 Also, $g \in \ker K_1(\varphi) \Rightarrow \exists v \in U_{\infty}(A)$ s.t. $g = [v]$, and $\tilde{\varphi}(v) = 1$.

Lemma. $\exists v \in U_{\infty}(I)$ s.t. $\tilde{\varphi}(v) = u$.

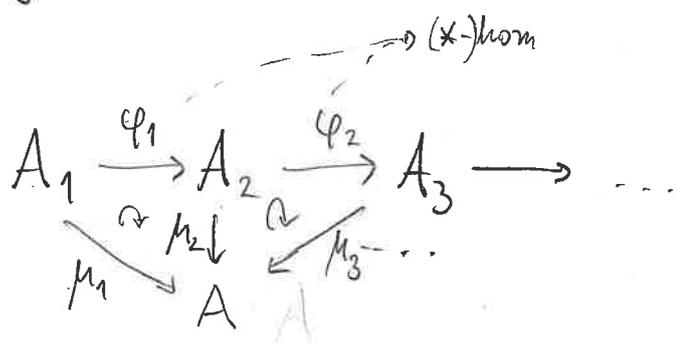
$\therefore [v] \in K_1(I)$ and $K_1(\varphi)([v]) = [\tilde{\varphi}(v)] = [u] = g$.
 $\Rightarrow \ker K_1(\varphi) \subseteq \text{Im } K_1(\varphi)$.

maybe just a remark!

Prop (K_1 is split-exact) $0 \rightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \rightarrow 0$ split exact.
 $\Rightarrow 0 \rightarrow K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B) \rightarrow 0$ split exact.

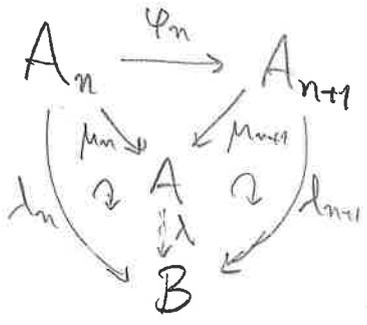
Prop (K_1 preserve direct sums) $K_1(A \oplus B) \cong K_1(A) \oplus K_1(B)$.

Digression - Inductive Limits (in Ab and C*-alg).



] inductive seq.
] $(A, \{\mu_n\}) =: \text{system}$
] inductive limit

Universal prop.



$\forall (B, \{\lambda_n\}) \text{ system}$
 $\exists! \lambda: A \rightarrow B \dots$

Notation: $A = \varinjlim A_n$.

Idea of construction

Denote

$\varphi_{mn} = \varphi_{m-1} \circ \dots \circ \varphi_n \quad (m > n)$
 $\varphi_{nn} = \text{id}_{A_n}$

$A_i := \{ (a_n) \in \prod_{i \geq 1} A_i \mid \exists n_0 \in \mathbb{N} \text{ st. } n \geq n_0 \Rightarrow a_n = \varphi_{nn_0}(a_{n_0}) \}$ "predictable tails"

with

$\mu_n: A_m \rightarrow A$
 $a \mapsto (a_n) \text{ with } a_n = \begin{cases} 0, & n < m; \\ \varphi_{nm}(a), & n \geq m. \end{cases}$

In C*-Alg, $\alpha((a_n)) := \limsup_{n \geq n_0} \|\varphi_{nn_0}(a_{n_0})\|$: \mathbb{C}^+ -seminorm.

Define $\varinjlim A_n := \overline{A/N_\alpha}$, where $N_\alpha = \{ (a_n) \in A : \alpha((a_n)) = 0 \}$.

Ex ① $A \xrightarrow{L_1} M_2(A) \xrightarrow{L_2} M_3(A) \rightarrow \dots \xrightarrow{\lim} M_n(A) \cong A \otimes K$
 $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

compt op. of a
 \downarrow
 separable Hilb. sp.

② $\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \dots \xrightarrow{\lim} \mathbb{Z} \cong \mathbb{Q}$.

Def An AF-algebra is a C^* -alg that is the inductive limit of finite dimensional C^* -alg.
 $i \in (0,1)$

Prop (K_i is continuous) $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \rightarrow \dots$ ind. seq.
 $\Rightarrow K_i(\varinjlim A_n) = \varinjlim K_i(A_n)$.

Prop $K_i(A) \cong K_i(M_n(A))$ and $K_i(A) \cong K_i(A \otimes K)$ with isomorphisms induced by $\lambda_n: A \rightarrow M_n(A)$ and $\varphi: A \rightarrow A \otimes K$.
 $a \mapsto \text{diag}(a, \dots, 0)$ and $a \mapsto (L_n(a))$

Ex $K_1(K) \cong K_1(\mathbb{C}) = 0$.
 $\cong K \otimes \mathbb{C}$

Coro $A \otimes K \cong B \otimes K \Rightarrow K_i(A) \cong K_i(B)$.
 (stably isomorphic)

Thm (Elliott) An unital AF-alg A is "completely determined" by $(K_0(A), K_0(A)^+, [1_A]_0)$.

$\hookrightarrow \{[p]_0 : p \in P_{\text{sa}}(A)\}$ "positive cone"

Remark (8.3) $A = C(X)$, X cpt. HFF.

$\Rightarrow K_1(C(X)) \cong \pi^1(X) \oplus \text{Ker}(\Delta)$ for a "nice" map $\Delta: K_1(C(X)) \rightarrow \pi^1(X)$,

Example: $\pi^1(\mathbb{T}) \cong \mathbb{Z}$

where $\pi^1(X) = U(C(X)) / U_0(C(X))$

In particular, $K_1(C(\mathbb{T})) \neq 0$.

\hookrightarrow cohomology

defined using a determinant.