# Chapter 2 - Projections and Unitary Elements

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Editor's Note: The statements of the theorems and definitions have been ~~over~~simplified. If this leads to lack of clarity, please see the book ☹.

Notation: $A$ $C^*$-algebra

when $A$ is unital!
$$\mathcal{U}(A) := \{u \in A : u^*u = uu^* = 1\}$$ "set of unitaries"
$$GL(A) := \{a \in A : a \text{ is invertible}\}$$
$$A_{sa} := \{a \in A : a \text{ is self-adjoint}\}.$$

## §2.1 - Homotopy classes of unitary elements

**Def** $X$ top. space, $a, b \in X$. We say that $a \sim_h b$ ($a$ is homotopic to $b$ in $X$) if $\exists$ cont. function $v : [0,1] \to X$ s.t. $v(0) = a$ and $v(1) = b$.

Rmk: in this section, we will consider $X = \mathcal{U}(A)$, $GL(A)$. Also, we can say that $a$ is "path connected" to $b$. Needless to say, $\sim_h$ is an equivalence relation...

**Def** $\mathcal{U}_0(A) := \{u \in \mathcal{U}(A) : u \sim_h 1 \text{ in } \mathcal{U}(A)\}$

$GL_0(A) := \{a \in GL(A) : a \sim_h 1 \text{ in } GL(A)\}$.

**Rmk**: Given $a, b \in A$, we always have $a \sim_h b$ in $A$ using $t \mapsto (1-t)a + tb$. But if $a, b \in \mathcal{U}(A)$, this is no longer a homotopy. Moreover, we will omit the reference to $X$ when the context is clear.

**Rmk**: $\left.\begin{array}{l} u_1 \sim_h v_1 \\ u_2 \sim_h v_2 \end{array}\right\} \implies u_1 u_2 \sim_h v_1 v_2$

**Lemma 2.1.3.** $A$ unital $C^*$-alg, $u, v \in \mathcal{U}(A)$.

(i) $h \in A_{sa} \Rightarrow e^{ih} \in \mathcal{U}_0(A)$.

(ii) $sp(u) \neq \mathbb{T} \Rightarrow u \in \mathcal{U}_0(A)$.

(iii) $\|u - v\| < 2 \Rightarrow u \sim_h v$.

Curiosity: $\|u-v\| \leq 2 \quad \forall u, v \in \mathcal{U}(A)$

**Sketch of proof:** (i) Define $f \in C(sp(h))$ by $f(s) := e^{is}$, and notice that $f^* = f^{-1}$, i.e., $f$ is an unitary. By function calculus, $f(h) = e^{ih} \in C^*(h, 1) \subseteq A$ is also unitary.

Moreover, define $f_t : sp(h) \to \mathbb{T}$ by $f_t(s) := e^{its}$. Since $t \mapsto f_t$ is continuous, so is $t \longmapsto f_t(h)$. Therefore,

$$e^{ih} = f_1(h) \sim_h f_0(h) = 1.$$

(ii) We use that $sp(u) \neq \mathbb{T}$ to define a "logaritm" and show that $u = e^{ih}$ for some $h \in A_{sa}$, then we apply (i).

(iii) $\|u - v\| < 2 \Rightarrow -1 \notin sp(v^* u) \overset{(ii)}{\Rightarrow} v^* u \sim_h 1 \Rightarrow u \sim_h v$. ∎

**Corollary 2.1.4.** $\mathcal{U}(M_n(\mathbb{C})) = \mathcal{U}_0(M_n(\mathbb{C}))$.

↳ this means $\mathcal{U}(M_n(\mathbb{C}))$ is path connected!

**Proof:** Every $u \in \mathcal{U}(M_n(\mathbb{C}))$ has finite spectrum, in other words $sp(u) \neq \mathbb{T}$ + Lemma 2.1.3 (ii). ∎

→ Important for the def. of $K_1$

**Lemma 2.1.5 (Whitehead)** $A$ unital $C^*$-alg, $u, v \in \mathcal{U}(A)$.
Then, $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & uv \end{pmatrix}$ in $\mathcal{U}(M_2(A))$.
In particular, $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathcal{U}(M_2(A))$.

**Proof:** $sp\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \neq \mathbb{T} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Hence,

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h$$

$$\sim_h \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}.$$

■

**Proposition 2.1.6/7** $A, B$ unital $C^*$-alg, $\varphi : A \to B$ surj. $*$-hom.

(i) $\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$.

(ii) $\mathcal{U}_0(A)$ is clopen relative to $\mathcal{U}(A)$.

(iii) $u \in \mathcal{U}_0(A) \iff u = e^{ih_1} \cdots e^{ih_n}$ for some $h_1, \ldots, h_n \in A_{sa}$.

(iv) $\varphi(\mathcal{U}_0(A)) = \varphi(\mathcal{U}_0(B))$.

(v) $\forall u \in \mathcal{U}(B)$ $\exists v \in \mathcal{U}_0(M_2(A))$ s.t. $\varphi_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$,
where $\varphi_2 : M_2(A) \longrightarrow M_2(B)$ is induced by $\varphi$.

(vi) $u \in \mathcal{U}(B)$, $u \sim_h \varphi(v)$ for some $v \in \mathcal{U}(A)$ $\Rightarrow$
$\Rightarrow u \in \varphi(\mathcal{U}(A))$.

$UU^*U = U$      Hilbert sp.

**Rmk (Polar decomposition).** Given $a \in B(\mathcal{H})$, there exists a unique partial isometry $U \in B(\mathcal{H})$ s.t.
$$a = U|a| \quad \text{and} \quad \ker(a) = \ker(U).$$
$|a| = (a^*a)^{1/2}$.

We don't have polar decomposition for any $C^*$-alg $A$, but if $a \in GL(A)$ we have a similar result!

**Proposition 2.1.8** $A$ unital $C^*$-alg.

(i) $z \in GL(A) \Longrightarrow |z| \in GL(A)$ and $z|z|^{-1} \in \mathcal{U}(A)$.

(ii) The map $\omega : GL(A) \to \mathcal{U}(A)$, $z \longmapsto z|z|^{-1}$ is cont., $\omega|_{\mathcal{U}(A)} = id_{\mathcal{U}(A)}$ and $\omega(z) \sim_h z$ in $GL(A)$. $(\forall z)$.

(iii) $u, v \in \mathcal{U}(A)$, $u \sim_h v$ in $GL(A) \Rightarrow u \sim_h v$ in $\mathcal{U}(A)$.

**Rmk:** Notice that $z = \omega(z)|z|$ is just a special case of the polar decomposition, in particular, we will use the letter $u$ instead of $\omega(z)$, i.e.,
$z = u|z|$.

Moreover, item (iii) allow us to "cheat" when proving results about homotopies in $\mathcal{U}(A)$.

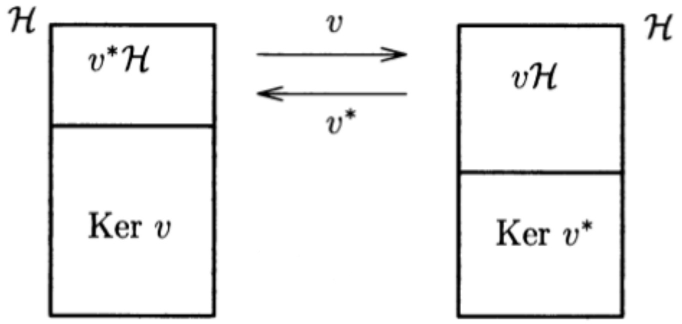
**Rmk (Carl Neumann series)** $A$ unital $C^*$-alg.
If $\|1-a\| < 1$, then $a$ is invertible and
$$a^{-1} = \sum_{n=0}^{\infty} (1-a)^n.$$

Analogously, $\|a\| < 1 \Rightarrow (1-a)^{-1} = \sum_{n=0}^{\infty} a^n \in GL(A)$.

**Proposition 2.1.11** $A$ unital $C^*$-alg, $a \in GL(A)$, $b \in A$.
$$\|a-b\| < \frac{1}{\|a^{-1}\|} \Rightarrow b \in GL(A) \text{ and } a \sim_h b \text{ in } GL(A).$$

## §2.2 - Equivalence of projections

Rmk: $v \in A$ is partial isometry if $vv^*v = v$ ($\Leftrightarrow v^*vv^* = v^*$).
If $A \subseteq B(\mathcal{H})$ then $v$ is maps $v^*\mathcal{H}$ isometrically to $v\mathcal{H}$.




We say that $v^*v$ is the support projection and $vv^*$ is the range projection.

Notation: $\mathcal{P}(A) := \{ p \in A : p^* = p = p^2 \}$ "set of projections".

Def. $p, q \in \mathcal{P}(A)$.

(i) (Murray-von Neumann equiv.)
$p \sim q$ if $\exists\, v \in A$ s.t $p = v^*v$ and $q = vv^*$.

(ii) (Unitary equiv.)
$p \sim_u q$ if $\exists\, u \in \mathcal{U}(\tilde{A})$ s.t. $q = upu^*$.

Rmk: If $p = v^*v$ and $q = vv^*$, this means that $v$ is a partial isometry. Moreover,
$$v = qv = vp = qvp.$$
Using this identities, we can show that $\sim$ is transitive:
Suppose $p \sim q$ and $q \sim r$ with
$p = v^*v$ ; $q = vv^* = w^*w$ and $r = ww^*$.
Consider $z = wv$, then
$$z^*z = v^*(w^*w)v = v^*(qv) = v^*v = p \quad \text{and}$$
$$zz^* = wvv^*w^* = wqw^* = ww^* = r.$$

**Proposition 2.2.2** $p,q \in P(A)$, $A$ unital $C^*$-alg. TFAE:

(i) $p \sim_u q$.

(ii) $q = upu^*$ for some $u \in \mathcal{U}(A)$.

(iii) $p \sim q$ and $1_A - p \sim 1_A - q$.

**Lemma 2.2.3** $p \in P(A)$, $a \in A_{s.a}$, $\delta := \|p-a\|$. Then,

$$sp(a) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta].$$

↳ Moral: if $a$ is "almost" a projection, the same holds for its spectrum.

Curiosity: $\|p-q\| \leq 1 \quad \forall p,q \in P(A)$

**Proposition 2.2.4** $p,q \in P(A)$.

$$\|p-q\| < 1 \implies p \sim_h q.$$

**Proof:** Consider $a_t := (1-t)p + tq$, $t \in [0,1]$. Then $a_t \in A_{sa}$,

$$\min\{\|a_t - p\|, \|a_t - q\|\} \leq \frac{\|p-q\|}{2} < \frac{1}{2}$$

and $t \mapsto a_t$ is cont.

Moreover $a_t \in \Omega_K = \{a \in A_{s.a} : sp(a) \subseteq K\}$ where $K = [-\delta, \delta] \sqcup [1-\delta, 1+\delta]$ and $\delta = \|p-q\|/2 < 1/2$.

↳ disjoint

Define $f: K \to \mathbb{C}$ by $f|_{[-\delta,\delta]} \equiv 0$ and $f|_{[1-\delta,1+\delta]} \equiv 1$.

Since $f^2 = \bar{f} = f$, each $f(a_t)$ is a projection and $t \mapsto f(a_t)$ is continuous. Thus,

$$p = f(p) = f(a_0) \sim_h f(a_1) = f(q) = q \quad \text{in } P(A)$$

∎

**Proposition 2.2.5** $A$ unital $C^*$-alg, $a, b \in A_{sa}$, $a = zbz^{-1}$ for some $z \in GL(A)$ and $z = u|z|$ polar decomp. $\Longrightarrow b = uau^*$.

The book as a "if and only if".

**Proposition 2.2.6.** $p, q \in P(A)$.
If there exists $u \in \mathcal{U}_0(\tilde{A})$ s.t. $q = upu^*$, then $p \sim_h q$.

**Proof:** Given $u \in \mathcal{U}_0(\tilde{A})$ with $q = upu^*$, consider a path $t \mapsto u_t \in \mathcal{U}_0(\tilde{A})$ s.t. $1 = u_0 \sim_h u_1 = u$.

Since $A \subseteq \tilde{A}$ is an ideal, $t \longmapsto u_t p u_t^*$ is a cont. path of projections from $p$ to $q$, i.e., $p \sim_h q$. ■

**Proposition 2.2.7.** $p, q \in P(A)$.
(i) $p \sim_h q \Rightarrow p \sim_u q$. Moreover, we can choose $u \in \mathcal{U}_0(\tilde{A})$.

(ii) $p \sim_u q \Rightarrow p \sim q$.

**Proof:** (i) If $p \sim_h q$, then $\exists\, p = p_0, p_1, \dots, p_n = q$ s.t. $\|p_{j+1} - p_j\| < \frac{1}{2}$ $\forall j = 0, \dots, n-1$.
Therefore it suffices to prove the statement if $\|p-q\| < \frac{1}{2}$.

Define $z := pq + (1-p)(1-q)$ and notice that $z \in \tilde{A}$, $pz = pq = zq$, and $\|z - 1\| = \dots < 2\|p-q\| < 1$.

$\xrightarrow{\text{Prop 2.1.11}} z \in GL(\tilde{A})$ and $z \sim_h 1$ in $GL(\tilde{A})$.

Given the polar decomposition $z = u|z|$.

$\xrightarrow{\text{Prop 2.2.5}} p = uqu^*$, i.e., $p \sim_u q$.

Moreover, $u \sim_h z \sim_h 1$ in $GL(\tilde{A}) \xrightarrow{\text{Prop 2.1.8}} u \sim_h 1$ in $\mathcal{U}(\tilde{A})$

$(u \in \mathcal{U}_0(A)!)$

$\triangleright A \trianglelefteq \tilde{A}$ is an ideal

(ii) If $upu^* = q$ for some $u \in \mathcal{U}(\tilde{A})$, then $v := up \in A$ and $v^*v = p^*u^*up = p^*p = p$

$vv^* = upp^*u^* = upu^* = q$. ∎

**Proposition 2.2.8** $p, q \in \mathcal{P}(A)$.

(i) $p \sim q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.

(ii) $p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.

Sketch of

**Proof:** (i) Given $v \in A$ s.t. $p = v^*v$ and $q = vv^*$, define

$$u := \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \text{ and } w := \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}.$$

One can verify that $wu \in M_2(A)$, even we only have $u, w \in M_2(\tilde{A})$. Moreover, $wu \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} (wu)^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, as desired.

(ii) Suppose $q = upu^*$, with $u \in \mathcal{U}(\tilde{A})$. By Whitehead's Lemma $\exists$ path $t \mapsto w_t$ s.t.

$$w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } w_1 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

Define $e_t := w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*$ and notice that $e_t \in \mathcal{P}(M_2(A))$.

Also, $t \mapsto e_t$ is cont. and

$$e_0 = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} = e_1.$$ ∎

**Example** ($p \sim q \not\Rightarrow p \sim_u q$).

Consider $A = B(\ell^2\mathbb{N})$ and $s : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$, $\delta_n \mapsto \delta_{n+1}$ the unilateral shift. Notice that $s^*s = 1$, i.e., $s$ is an isometry.

Moreover, $s^*s \sim ss^*$ by definition.

On the other hand, $0 = 1 - s^*s \nsim 1 - ss^* \neq 0$.
Therefore, $s^*s \nsim_u ss^*$ by Prop. 2.2.2.

Rmk: An example of why $p \sim_u q \not\Rightarrow p \sim_h q$ is more complex.

## §2.3 - Semigroups of Projections

Def. $P_n(A) := P(M_n(A))$, $P_\infty(A) := \bigcup_{n=1}^{\infty} P_n(A)$.

Digression (Inductive Limits)
We can look at $P_\infty(A)$ as a subset of a $*$-algebra $M_\infty(A)$ defined in the following way:
Consider the (non-unital) injective $*$-homomorphisms

$$i_n : M_n(A) \hookrightarrow M_{n+1}(A)$$
$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Then one can think about $M_\infty(A) := \bigcup_{n=1}^{\infty} M_n(A)$ as the $*$-alg. of infinite matrices with only finitely many non-zero entries. In this sense, $P_\infty(A)$ is the set of projections of $M_\infty(A)$.

Moreover, one can define a $C^*$-norm on $M_\infty(A)$ and consider the completion. Then

$$\overline{M_\infty(A)} \cong A \otimes \mathbb{K}, \quad \text{(stabilization of } A\text{)}$$

where $\mathbb{K}$ is the $C^*$-alg. of compact operator of the separable Hilbert space $\mathcal{H}$.

The whole point of the stabilization is that we have "more" space and $A \otimes \mathbb{K} \cong M_n(A \otimes \mathbb{K})$. In particular, the three equiv. relations previously defined are the same!

**Def** Given $p, q \in P_\infty(A)$, we say that $p \sim_0 q$ if $\exists v \in M_\infty(A)$ s.t. $p = v^* v$ and $q = v v^*$.

Moreover, define an operation $\oplus$ on $P_\infty(A)$ by

$$p \oplus q = \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

**Proposition 2.3.2.** $p, q, r, q', r' \in P_\infty(A)$.

(i) $p \sim_0 p \oplus 0_n$, $\forall n \in \mathbb{N}$, where $0_n = 0_{M_n(A)}$.

(ii) $p \sim_0 p'$, $q \sim_0 q' \Rightarrow p \oplus q \sim_0 p' \oplus q'$.

(iii) $p \oplus q \sim_0 q \oplus p$.

(iv) $p, q \in P_n(A)$, and $pq = 0 \Rightarrow p + q \in P_n(A)$ and $p + q \sim_0 p \oplus q$.

(v) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

**Def** Define $\mathcal{D}(A) := P_\infty(A)/\sim_0$ and denote by $[p]_\mathcal{D}$ the equiv. class containing $p$.
Define addition on $\mathcal{D}(A)$ by

$$[p]_\mathcal{D} + [q]_\mathcal{D} := [p \oplus q]_\mathcal{D}.$$

**Rmk:** Prop 2.3.2 implies that $(\mathcal{D}(A), +)$ is an abelian semigroup.