Rational Points Reading Group: Faithfully Flat Descent

Guiding Principle / Question:

Suppose we want to construct a k-variety, X. However, we might not be able to do this directly, and instead we may only be able to construct a k'-variety X' for k'/k a fld. extension.

Question: Is X' then the base extension of some k-variety X? If so, how to construct X?

This is a special case of descent.

4.1. Gluing Sheaves

Let S be a Top. Space w/ $\{S_i\}$ an open cover.

Suppose $\mathcal{F}_i$ a sheaf on S_i $\forall i$.

Under what conditions does $\exists$ a sheaf $\mathcal{F}$ on S with $\mathcal{F}|_{S_i} \cong \mathcal{F}_i$?

If $\mathcal{F}$ exists, then the restrictions of

$\mathcal{F}|_{S_i}$ and $\mathcal{F}|_{S_j}$ to $S_{ij} := S_i \cap S_j$ must be isomorphic.

So we should insist that $\forall i,j$ $\exists$ isomorphism

$$\phi_{ij} : \mathcal{F}_i|_{S_{ij}} \to \mathcal{F}_j|_{S_{ij}}$$

Can we then "glue" the $\mathcal{F}_i$ via ϕ_{ij}?

Consider on $S_{ijk} := S_i \cap S_j \cap S_k$

The sheaves $\mathcal{F}_i|_{S_{ijk}}, \dots, \mathcal{F}_k|_{S_{ijk}}$ are identified in pairs by $\phi_{ij}, \phi_{jk}, \phi_{ik}$ forming a triangle of identifications.

We should insist that two sides of the triangle should give the same identification as the third:

$$\forall i,j,k \quad \phi_{jk} \circ \phi_{ij} = \phi_{ik} \text{ on } S_{ijk}.$$

"Cocycle Condition".

The gluing theorem states that if we have sheaves $\mathcal{F}_i$ on S_i, and if there exist ϕ_{ij} satisfying the above 2 conditions, then up to isomorphism,

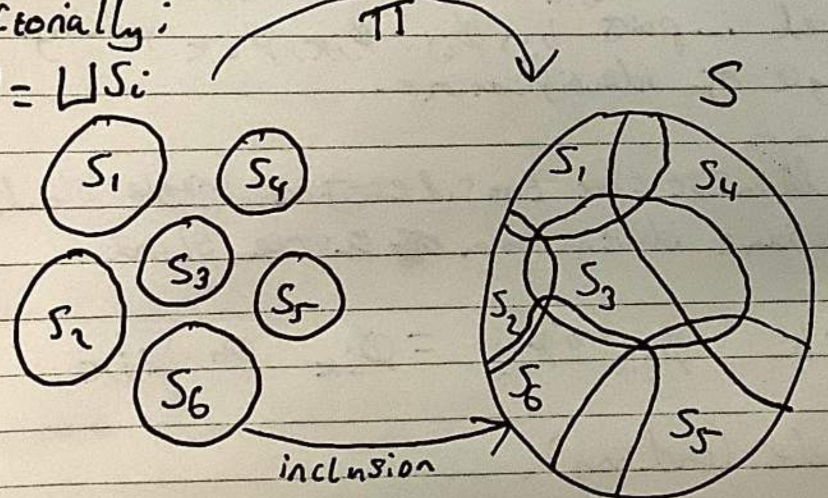
$\exists$ a unique sheaf $\mathcal{F}$ on S with isomorphisms $\phi_i : \mathcal{F}|_{S_i} \to \mathcal{F}_i$ s.t. ϕ_i, ϕ_j identify on $\mathcal{F}_{S_{ij}}$ with ϕ_{ij}.

It turns out there is a nice re-formulation of the gluing conditions!

Let $S' = \bigsqcup_i S_i$ and $\pi : S' \to S$ be the "open covering morphism" s.t. π is the inclusion on each S_i

Pictorially:

$S' = \bigsqcup S_i$



To give a sheaf $\mathcal{F}_i$ $\forall i$ is equivalent to giving a single sheaf $\mathcal{F}'$ on S'.

Question: does $\exists$ a sheaf $\mathcal{F}$ on S s.t.

the sheaf $\pi^{-1}\mathcal{F}$ on S' is isomorphic to $\mathcal{F}'$?

Let $S'' := S' \times_S S'$

This is equal to the disjoint union of the $S_i \times_S S_j = S_i \cap S_j =: S_{ij}$.

Let $p_1 : S'' \to S'$
$p_2 : S'' \to S'$ be the two projections.

The sheaf $p_1^{-1}\mathcal{F}'$ restricted to the piece indexed by ij corresponds to the sheaf $\mathcal{F}_i|_{S_{ij}}$.

Thus: asking for isomorphisms ϕ_{ij} $\Leftrightarrow$ asking that we are given an isomorphism $\phi : p_1^{-1}\mathcal{F}' \to p_2^{-1}\mathcal{F}'$ of sheaves on S''.

Let $S''' := S' \times_S S' \times_S S'$, let p_{12} be the projection onto the 1st and 2nd coords etc.

Then $p_{12}^{-1}\phi$ is an isomorphism of sheaves on S'''

The cocycle condition becomes:

$$p_{13}^{-1}\phi = p_{23}^{-1}\phi \circ p_{12}^{-1}\phi$$

(Skip section 4.2 on FFD for QC sheaves)

4.3. FFD for Schemes

Idea: In place of the Zariski open covering morphisms $S' \to S$, we can use the more general fpqc morphisms of schemes.

Let $p : S' \to S$ be fpqc, define S'', S''' as before but use fibre product of schemes. Define projection maps as before.

Let X' be an S'-scheme. Under what conditions is $X' \cong p^* X$ for some S-scheme X? (Notation: $p^* X = X \times_S S'$)

Descent Data

A descent datum on an S'-scheme X' is an S''-isomorphism $\phi : p_1^* X' \to p_2^* X'$ satisfying the usual cocycle condition

$$p_{13}^* \phi = p_{23}^* \phi \circ p_{12}^* \phi$$

The pairs (X', ϕ) are objects in a category (of S'-schemes with descent data).

If X is an S-scheme, then $p^* X$ has canonical descent datum ϕ_X. Call ϕ effective if $(X', \phi) \cong (p^* X, \phi_X)$ for some S-scheme X.

Open subschemes stable under a descent datum

Let X' be an S'-scheme with descent datum $\phi : p_1^* X' \to p_2^* X'$. An open subscheme $U' \subseteq X'$ is called stable under ϕ if ϕ induces a descent datum on U', that is, if ϕ restricts to an isomorphism $p_1^* U' \to p_2^* U'$ of S''-schemes.

The Descent Theorem for Schemes

Def^n A morphism $f : X \to S$ is affine if $f^{-1} S_0$ is affine $\forall$ open affine subschemes S_0 of S. In this case we call X an affine S-scheme.

Warning! affine S-scheme $\not\Rightarrow$ affine scheme!

Better name might be "relatively affine".

Def^n A scheme is quasi-affine if it is an open subscheme of an affine scheme and it's quasi compact.

A morphism $f: X \to S$ is <u>quasi-affine</u> if $f^{-1}S_0$ is quasi-affine $\forall$ affine open subscheme S_0 of S.

<u>Theorem 4.3.5.</u>

Let $p: S' \to S$ be a fpqc morphism of schemes

i) The Functor $X \mapsto p^* X$ from $\{S\text{-schemes}\}$ to $\{S'\text{-schemes with descent data}\}$ is fully faithful.

ii) The Functor $X \mapsto p^* X$ from $\{\text{quasi-affine } S\text{-schemes}\}$ to $\{\text{quasi-affine } S'\text{-schemes with descent data}\}$ is an equivalence of categories.

iii) Suppose S, S' are affine. Then a descent datum ϕ on an S'-scheme X' is effective iff X' can be covered by quasi-affine open subschemes which are stable under ϕ.

NB// ii), iii) also hold if quasi-affine is replaced by affine everywhere.

<u>Descending Properties of Morphisms</u>

When an S-scheme X is base extended to X' an S'-scheme, we know that X' inherits many properties of X. Conversely, one would hope that when an S'-scheme X' is descended to an S-scheme X that X inherits properties of X'.

This is true for many properties under fpqc descent.

<u>Theorem:</u>

Let "adjective" be a property (see p302-303 of Poonen for the list - the fpqc descent column)

Let $S' \to S$ be fpqc. $\forall$ S-scheme X let $X' = X_{S'}$.

i) If the base ext. $X' \to S'$ is "adjective", then the original morphism $X \to$ is "adjective".

ii) More generally, if $X \to Y$ is a morphism of S-schemes and its base extension $X' \to Y'$

by $S' \to S$ is "adjective", then the original morphism $X \to Y$ is "adjective":

<u>Pictorially:</u>

$$\begin{tikzcd}
X' \arrow[rrr, dashed] \arrow[dr, "adj."] \arrow[dd] & & & X \arrow[dr, "adj."] \arrow[dd] & \\
& Y' \arrow[rrr, dashed] \arrow[dl] & & & Y \arrow[dl] \\
S' \arrow[rrr, dashed, "fpqc"'] & & & S &
\end{tikzcd}$$

4.4. Galois Descent

We now study a particularly nice case of fpqc descent.

Let k a field, k' finite Galois extension of k.
$S = \operatorname{Spec} k$, $S' = \operatorname{Spec} k'$

Then $S' \to S$ is fpqc and we can apply Theorem 4.3.5. iii) to say something about descending k'-schemes to k-schemes.

Let $G = \operatorname{Gal}(k'/k)$. The left-action of G on k' induces a right action of G on S': each $\sigma \in G$ induces an automorphism σ^* of S'.

<u>Proposition 4.4.2.</u>

i) Giving a descent datum on a k'-scheme X' is equivalent to giving a right action of G on X' compatible with the right action of G on S', i.e. to giving a collection of isomorphisms $\tilde{\sigma} : X' \to X'$ for $\sigma \in G$ s.t.

$$\begin{tikzcd}
X' \arrow[r, dashed, "\tilde{\sigma}"] \arrow[d] & X' \arrow[d] \\
S' \arrow[r, "\sigma^*"] & S'
\end{tikzcd}$$

commutes $\forall \sigma \in G$, and $\widetilde{\sigma\tau} = \tilde{\tau}\tilde{\sigma}$ $\forall \sigma, \tau \in G$.

ii) An isomorphism between k'-schemes with descent data is a k'-isomorphism that is equivariant for the G-actions above.

iii) An open subscheme U' of a k'-scheme X'

is stable under a descent datum iff $\hat{\sigma}(U') = U'$ $\forall \sigma \in G$.

The morphisms $\hat{\sigma}$ are not morphisms of k'-schemes since they lie over σ^* and not id.

We could rewrite the conditions of Prop 4.4.2 in terms of k'-morphisms.

Recall: k'-schemes can be transformed by $\sigma \in G$ (see Section 2.2.)

<u>Prop 4.4.4.</u> Let X' be a k'-scheme.

i) Giving a descent datum on X' is equivalent to giving a collection of k'-isomorphisms $f_\sigma : {}^\sigma X' \to X'$ for $\sigma \in G$ satisfying the "cocycle condition" $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma(f_\tau)$ $\forall \sigma, \tau \in G$.

ii) An isomorphism between varieties with descent data, say X' with $(f_\sigma)_{\sigma \in G}$ and Y' with $(g_\sigma)_{\sigma \in G}$, is a k'-isomorphism $b : X' \to Y'$ s.t. $f_\sigma = b^{-1} g_\sigma ({}^\sigma b)$ $\forall \sigma \in G$.

iii) An open subscheme $U' \subseteq X'$ is stable under a descent datum described above iff $f_\sigma({}^\sigma U') = U'$ $\forall \sigma \in G$.

<u>Corollary 4.4.6.</u>

Let k'/k be a finite Gal. extension of fields. Let X' be a quasi-projective k'-scheme. Suppose that we are given k'-isomorphisms $f_\sigma : {}^\sigma X' \to X'$ $\forall \sigma \in G$ satisfying $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma(f_\tau)$ $\forall \sigma, \tau \in G$

Then $X' = X_{k'}$ for some k-scheme X

<u>Proof:</u>
Giving the f_σ $\Leftrightarrow$ giving a right-action of G on X'. By Theorem 4.3.5: it suffices to show that X' can be covered by G-invariant quasi-affine open subsets.

Fix an embedding $X' \hookrightarrow \mathbb{P}^n_{k'}$. Given $x' \in X'$ we can choose a hypersurface $H \subset \mathbb{P}^n_{k'}$ that does not meet the G-orbit of x'.

Let $U' = X' - H$. Then $\bigcap_{\sigma \in G} \hat{\sigma}(u')$ is a

quasi-affine open subset of X' containing x'!
It is also obviously G-invariant.
Since one can construct these sets $\forall x' \in X'$
we have a covering of X'.

4.5 Twists

X = quasi-projective k-variety

Def^n

A k'/k-twist (or k'/k-form) of X is a k-variety Y s.t. $\exists$ an isomorphism $\phi: X_{k'} \xrightarrow{\sim} Y_{k'}$

A twist of X is a k_s/k-twist of X.

The set of k-isomorphism classes of k'/k-twists of X is a pointed set with neutral given by the isomorphism class of X.

The action of G on k' induces an action of G on the Automorphism group $\mathrm{Aut}\, X_{k'}$

Theorem 4.5.2: There is a natural bijection of pointed sets

$$\frac{\{k'/k\text{-twists of } X\}}{k\text{-isomorphism}} \xrightarrow{\sim} H^1(G, \mathrm{Aut}\, X_{k'})$$

Warning! $\mathrm{Aut}\, X_{k'}$ maybe non-abelian so non-abelian cohomology theory may be needed!

e.g. (Exercise 4.3)
Let E be an elliptic curve over k: $y^2 = x^3 + ax + b$

Let $d \in k^\times \setminus k^{\times 2}$, $L = k(\sqrt{d})$
$\sigma \in \mathrm{Gal}(L/k)$ the non-trivial element.

Let E' be the elliptic curve over k with equation $dy^2 = x^3 + ax + b$

E' is a L/k twist of E, since they are isomorphic via the morphism:

$$\varphi: E'(L) \to E(L)$$
$$(x,y) \mapsto (x, y\sqrt{d}), \quad P \neq O$$
$$O \mapsto O$$

Claim:
As groups:
$$E'(k) \cong \hat{E}(L) := \{P \in E(L) : {}^{\sigma}P = -P\}$$

Let $t, u, v, w \in K$

$$\hat{E}(L) = \{(t+u\sqrt{d}, v+w\sqrt{d}) \in E(L) : \sigma(t+u\sqrt{d}, v+w\sqrt{d}) = (t+u\sqrt{d}, -v-w\sqrt{d})\} \cup \{O\}$$

$$= \{(t, w\sqrt{d}) \in E(L)\} \cup \{O\}$$

Clearly the map

$$E'(K) \to \hat{E}(L)$$
$$(x,y) \mapsto (x, y\sqrt{d})$$
$$O \mapsto O$$

is a rational map $\Rightarrow$ morphism
sends basepoint to basepoint $\Rightarrow$ isogeny
$\Rightarrow$ group homomorphism
It is also obviously bijective.

So $E'(K) \cong \hat{E}(L)$ as groups.

Severi – Brauer Varieties

Defn A Severi-Brauer (SB) variety over k is a twist of the k-variety $\mathbb{P}_k^{n-1}$ for some $n \geq 1$.

e.g. The 1-dim SB varieties over k are exactly the 'nice' genus 0 curves.

It is natural to use cohomology to study these.

Note: $\mathrm{Aut}\,\mathbb{P}^{n-1}_{k_s} = PGL_n(k_s) = \underbrace{\mathrm{Aut}(M_n(k_s))}$

Automorphism grp of the matrix algebra.

Applying the bijection theorem from above and recalling some stuff from section 1.5.5

$$\Rightarrow \{(n-1)\text{-dim SB varieties } /k\}$$
$$\updownarrow$$
$$H^1(G_k, PGL_n(k_s)) \hookrightarrow \mathrm{Br}\,k.$$
$$\updownarrow$$
$$\{n^2\text{-dimensional Azumaya } k\text{-algebras}\}$$

Remark:

One can show that $X_{\bar{K}} \cong \mathbb{P}^{n-1}_{\bar{K}} \Rightarrow X_{K_s} \cong \mathbb{P}^{n-1}_{K_s}$

Proposition 4.5.10.

TFAE: (For $(n-1)$-dim SB variety X over K)

i) $X \cong \mathbb{P}^{n-1}_K$
ii) X birational to $\mathbb{P}^{n-1}_K$
iii) $X(K) \neq \emptyset$

Proof: (Maybe skip if short on time)
i) $\Rightarrow$ ii) Trivial
ii) $\Rightarrow$ iii) Follows from Lang-Nishimura Theorem; see Corollary 3.6.16.
iii) $\Rightarrow$ i)
Pick $x \in X(K)$. Since X is SB, $\exists$ isomorphism $X_{K_s} \xrightarrow{\sim} \mathbb{P}^{n-1}_{K_s}$. Compose it with an automorphism of $\mathbb{P}^{n-1}_{K_s}$ s.t. $x \longmapsto P := (1:0 \cdots :0) \in \mathbb{P}^{n-1}_K$

Then (X, x) may be viewed as a twist of the pointed variety $(\mathbb{P}^{n-1}_K, P)$

The Aut.s of $(\mathbb{P}^{n-1}_K, P)$ over K_s are the Aut.s of

$\mathbb{P}^{n-1}_{K_s}$ fixing P

They form a subgroup of $PGL_n(K_s)$:

$$\mathrm{Aut}(\mathbb{P}^{n-1}_{K_s}, P) = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \bmod K_s^\times \right\}$$

$$\cong \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \right\}$$

The "forget the first row and column" map is a group homomorphism to $GL_{n-1}(K_s)$ and we obtain a G_K-equivariant exact sequence:

$$0 \to (K_s)^{n-1} \to \mathrm{Aut}(\mathbb{P}^{n-1}_{K_s}, P) \to GL_{n-1}(K_s) \to 0$$

By Prop 1.3.15 i) and iii), the H^1 of the groups at each end is trivial, so the middle $H^1(G_K, \mathrm{Aut}(\mathbb{P}^{n-1}_{K_s}, P))$ is trivial too. Equivalently by Theorem 4.5.2, $(\mathbb{P}^{n-1}_K, P)$ has no non-trivial twists. $\Rightarrow (X, x) \cong (\mathbb{P}^{n-1}_K, P)$

$\Rightarrow X \cong \mathbb{P}_K^{n-1}$

Theorem 4.5.11

SB varieties over Global fields Satisfy the local-to-global (Hasse) principle!

Proof:

Let X be a variety, x the corresponding Brauer Class in $\mathrm{Br}\,K$.

Let $n-1 = \dim X$. By previous Prop,

$$X(K) \neq \emptyset \Leftrightarrow X \cong \mathbb{P}_K^{n-1} \Leftrightarrow x = 0$$

The variety X_{K_v} is a SB variety over K_v corresponding to the image x_v of x in $\mathrm{Br}\,K_v$, so we Similarly have:

$$X(K_v) \neq \emptyset \Leftrightarrow x_v = 0$$

From the first talk (see also Section 1.5.a) the map $\mathrm{Br}\,K \to \bigoplus_v \mathrm{Br}\,K_v$ is injective

So we are done $\square$

4.6. Restriction of Scalars

Motivating Example:

Let $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{2})$, X the affine curve in $\mathbb{A}_L^2$ defined by $x_1 x_2 + (5 + 7\sqrt{2}) = 0$

What are the L-points on X?

Let $x_1 = x + y\sqrt{2}$ for $x, y, z, t \in \mathbb{Q}$
$x_2 = z + t\sqrt{2}$ and substitute in

$$\Rightarrow (xz + 2yt + 5) + (xt + yz + 7)\sqrt{2} = 0$$

$\Rightarrow$ define a hypersurface $\mathcal{X}$ in $\mathbb{A}_{\mathbb{Q}}^4$ by

$$xz + 2yt + 5 = 0$$
$$xt + yz + 7 = 0$$

So we see that $\mathcal{X}(\mathbb{Q}) = X(L)$

We have turned a problem concerning a "small" variety over a "big" field into a "big" variety over a "small" field.
This is a Special case of Restriction of Scalars.

More generally, let $L \supseteq k$ be a fin. ext.
X an L-variety.
Want to construct a k-variety $\mathcal{X}$ whose arithmetic "mimics" X over L.
ideally want a bijection $\mathcal{X}(k) \cong X(L)$
But this condition is not enough to determine $\mathcal{X}$ uniquely.

<u>Def^n</u> The <u>restriction of scalars</u> (Weil restriction)
$\mathcal{X} = \mathrm{Res}_{L/k}(X)$, if it exists, is a k-variety characterised by the existence of bijections:

$$\mathcal{X}(S) \to X(S \times_k L) = \mathrm{Hom}_L(S \times_k L, X)$$

for each k-scheme S, varying functorially in S
i.e., $\forall$ k-morphism $f: S \to T$, the diagram

$$\begin{array}{ccc} \mathcal{X}(T) & \longrightarrow & X(T \times_k L) \\ \downarrow & & \downarrow \\ \mathcal{X}(S) & \longrightarrow & X(S \times_k L) \end{array}$$

induced by f and its base extension $f_L: S \times_k L \to T \times_k L$

commutes.

If X is an affine L-variety (like in the example) then $\mathcal{X} := \mathrm{Res}_{L/k} X$ exists as an affine k-variety and can be constructed as follows:

Write $X = \mathrm{Spec}\, L[x_1, \dots, x_n]/(f_1, \dots, f_m)$

Choose a basis $e_1, \dots, e_s$ for L over k.
Introduce new variables y_{ij} with $1 \le i \le n$, $1 \le j \le s$ and substitute

$$x_i = \sum_{j=1}^{s} y_{ij} e_j \quad \forall\, i \text{ into } f_r \text{ for each } r$$

So that: $f_r(x_1, \dots, x_n) = F_{r,1} e_1 + \dots + F_{r,s} e_s$

for some polynomials $F_{r,l} \in k[\{y_{ij}\}]$.

Then $\mathcal{X} = \mathrm{Spec}\, k[\{y_{ij}\}]/(\{F_{r,l}\})$

This is exactly what we did in the example!

For non-affine varieties things aren't so nice.

<u>Prop</u>
Let $L \supseteq K$ as above, X an L-variety
If every finite subset of X is contained in some affine open subset of X, then $\mathrm{Res}_{L/K} X$ exists.

<u>Important Remark</u>
Any quasi-projective variety X over L satisfies the above Hypothesis.

Restriction of Scalars does the following:

$$\left\{ \begin{array}{l} \text{Questions about varieties} \\ \text{over larger fields} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Questions about} \\ \text{Higher Dimensional} \\ \text{Varieties over Smaller} \\ \text{fields} \end{array} \right\}$$

e.g. if L fin. separable ext. of global field K, A an abelian variety over $L \Rightarrow$ BSD holds for A over L iff it holds for $\mathrm{Res}_{L/K} A$ over K.