Typical Elements

<u>Recap</u> F_m free group. $S_l \subset F_m$ set of <u>reduced</u> words of length l. Let $0 \leq d \leq 1$ be the density.

A random set of relators of length l at density d is a $(2m-1)^{dl}$ – tuple of elements of S_l.

(Uniformly and independently).

A random group, density d, and length l is group G presented by $\langle a_1, \dots, a_m \mid R \rangle$ ← random set of relators

(Phase transition)

<u>Thm 11</u> – Let G be a random group at density d.

- If $d < \frac{1}{2}$ with overwhelming probability G is infinite hyperbolic, torsion-free and ...
- If $d > \frac{1}{2}$ then " " " either $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$.

Robustness – ~~Taken~~ Random quotient of group with property P has property P.

Theorem 38) Let G_0 be a non-elementary, torsion-free hyperbolic group, S finite generating set. Let B_ℓ be set of elements of G_0 with norm at most ℓ.

$B_\ell = \{ g \in G_0 \mid |g|_S \le \ell \}$

Let $0 < d < 1$. Let $R \subset G_0$ be $(\# B_\ell)^d$ picked elements from B_ℓ. $G := G_0 / \langle R \rangle$

- If $d < \frac{1}{2}$ with over. prob G is (non-elementary) hyperbolic
- If $d > \frac{1}{2}$ then with O P $G = \{e\}$

p 72 "Not clear whether a random quotient of a torsion free hyperbolic group is torsion free. But geometric dimension 2 is preserved => torsion free

Typical elements II

Growth of random quotients does not change much.

Recall $g = \lim_{L \to \infty} \frac{1}{L} \log_{2m-1} \# B_L$ (↗ growth exponent)

where $G = \langle a_1, \dots, a_m \mid R \rangle$

Note: If G is a random group of density $> \frac{1}{2}$ then $g \to 0$

Thm 25) Let $\varepsilon > 0$, $0 \le d < \frac{1}{2}$ then with OP the growth exponent of a random group at d lies in $[1-\varepsilon, 1)$

Back to Random quotients (Thm 39) G_0 non-elementary, torsion-free hyperbolic group with f.g. set S.
Let g be growth exponent of G_0
Let $d < \frac{1}{2}$. Let G be random ~~elements~~ quotient. Then, $\forall \varepsilon > 0$ with OP. the growth exponent of G is in $(g - \varepsilon, g)$.

Random quotients by words. Instead of quotienting by a random set of elements, we quotient by random words from the generators.

"Advantage that we are independent of the initial group"

Recap: G with generating set S. A random walk is a chosen sequence of elements starting with e and at each step multiplying by $s \in S \cup S^{-1}$.

p_t is probability at time t that the random walk has returned to e.

Note $p_{t+t'} \geq p_t p_{t'}$

$$\rho = \lim_{\substack{t \to \infty \\ t \text{ even}}} (p_t)^{1/t}$$

Fact. $\frac{\sqrt{2m-1}}{m} \leq \rho \leq 1$

$\rho = 1$ iff G amenable

$\rho = \frac{\sqrt{2m-1}}{m}$ iff $G \cong F_m$

If G is random group, density d.

If $d > \frac{1}{2}$ then $\rho = 1$.

Thm 25: "ρ does not depend on density if $d < \frac{1}{2}$"

then $\rho(G) \in (\rho(F_m), \rho(F_m) + \varepsilon)$ $\varepsilon > 0$.

Typical elements III

<u>Thm 40</u> Let G_0 be torsion-free hyperbolic gen. by $a_1, \dots, a_m$. Let $(w_t)_{t \in \mathbb{N}}$ be [illegible] random walk.

$$d_{crit} := -\log_{2m} \rho(G_0)$$

Note $d_{crit} > 0$ unless G_0 is elementary.

Let $0 \le d \le 1$ and let W_ℓ be all $(2m)^\ell$ words of length ℓ. Let R be random set by picking $(2m)^{d\ell}$ times a random word in W_ℓ.

$G = G_0/\langle R \rangle$ then with o.p.

$d < d_{crit}$ implies G is (non-ele) hyperbolic

$d > d_{crit}$ then $G = \{e\}$

Thm 42: G_0 as before. Let G be quotient by random words at [illegible] density d. Then $\forall \varepsilon > 0$ w.o.p. [illegible] ε.

<u>Cor</u> The critical density of [illegible] [illegible] [illegible] of G

Iteration <u>Pbm</u> Let F_m free group (non-cyclic) [illegible]. Let $d_1 := -\log_{2m} \rho(F_m)$ and [illegible] set R_ℓ [illegible] d of random words of length ℓ. Let [illegible] and $G = F_m/\langle R \rangle$. Then [illegible] [illegible] [illegible] [illegible] G [illegible] [illegible]

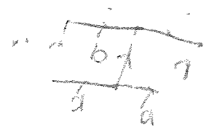
[illegible] [illegible] non-elementary hyperbolic [illegible]

Van Kampen

F_{M} [illegible] $= G$ [illegible]

Then [illegible] together

[illegible] 

and [illegible]  [illegible] σ^2 [illegible]

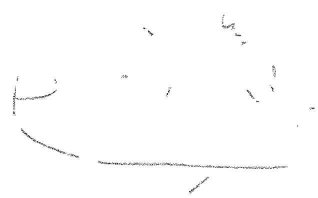
Thus [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Length [illegible] 

Let [illegible]

[illegible] Want [illegible]

[illegible] new length L [illegible]

[illegible] $= e$ [illegible] $(2m)^{-(1-\theta)}$ [illegible]

[illegible]