Random Groups

Schedule:
1) Motivation
2) Properties $\to$ subgroups, presentations
3) Randomness
4) Gromov's density model.

1) Motivation

- What properties does a typical group have?
  $\hookrightarrow$ elements, subgroups, presentations.

Fact: almost all groups are hyperbolic.

2) Discrete representations

- Subgroups being free / words reduced.

Notation: - R_n = reduced words length n
- Cyclically reduced if u^2 is also reduced.
- u reduced, $\exists!\, v, w$ st w c-red and $u = v^{-1} w v$.

- Subgroups of $F(A)$

Notation: $\vec{h} = (h_1, \dots, h_k)$, $\to \langle \vec{h} \rangle$, $\to \langle\langle \vec{h} \rangle\rangle$

- Quasi-convexity of subgroups of G.

Definition: × $H \leq G$ is quasi-convex if $\exists k \geq 0$ if for every geodesic word $u = a_1 \dots a_n$ st $u \in H$ $\exists v_i$ of length $\leq k$ such that $a_1 \dots a_i v_i \in H$

× $Y \subseteq X$ is quasi-convex if when $k \geq 0$ st any geodesic connecting a pair of points in Y is in the k-nbh of Y.

Fact: Every f.g. subgroup of a free group is free and quasi-convex.

Malnormality: $g^{-1}Hg \cap H = \phi \quad \forall g \notin H$.

almost if $g^{-1}Hg \cap H$ is finite.

Purity: $x^n \in H \Rightarrow x \in H \quad \forall n \in \mathbb{N}, \forall x$ (infinite order)

almost pure

Fact: * It is decidable whether a f.g. subgroup of F_n is malnormal or pure.

* Almost malnormal is decidable for quasi-convex subgroups of hyperbolic groups.

Properties of presentations: combinatorial or geometric

The $C'(\lambda)$ small cancellation property.

Definition: A piece in a tuple $\vec{h}$ of cyclically reduced words is a word u that occurs as a prefix of two distinct elements of $\vec{h}$ or its inverses.

Example: $\vec{h} = (a^{-1}a^{-1}bb,\ b\,a\,b^{-1}a\,b,\ a^{-1}ba^{-1}b)$

$$u = ab^{-1}a$$

A finite presentation $\langle A \mid \vec{h} \rangle$ has property $C'(\lambda)$ if any piece u in $\vec{h}$ satisfies $|u| < \lambda |h_i| \ \forall i$ st u is a piece of h_i.

$\rightsquigarrow$ Why do we care? $C'(\leq \frac{1}{6}) \Rightarrow$ hyperbolicity.

Randomness

Few relator model:

$R_{k,l}$ set of all possible presentations with k relators of length at most l.
P: property.

<u>Definition</u>: P occurs with overwhelming probability in the model if the share of presentations in $R_{k,l}$ which have property P tends to 1 as $l \to \infty$.

<u>Proposition</u>: $\to \forall k \ \forall \lambda > 0$ the $C'(\lambda)$ occurs with overwhelming probability in the few relator model. So does hyperbolicity therefore.
$\to \forall k$ torsion-free and 2-dimensional.

<u>Gromov's Density Model</u>

<u>Definition</u>: F_m free group, $m \geq 2$.
$\forall l \in \mathbb{Z}$, let $S_l \subseteq F_m$ to be the set of reduced words of length l.
<u>Density</u>: $0 \leq d \leq 1$ A random set of relators at density d at length l is a $(2m-1)^{dl}$-tuple of elements in S_l.

A <u>random group</u> at density d at length l is the group $G = \langle a_1, \dots, a_m \mid R \rangle$ where R is a random set of relators at density d at length l.
A property happens with overwhelming probability at density d if $\mathbb{P}(\text{occurring}) \longrightarrow 1$ as $l \to \infty$ for a fixed d.

Proposition: R, d, ℓ, $0 \le \alpha \le d$ with o.p. any reduced word of length $\alpha \ell$ appears as a subword in R.

e.g.: Set of $2^{d\ell}$ random words of length ℓ in a, b
$\simeq d\ell$ 'a's

Proposition: ① $\lambda > 0$, $d < \lambda/2$, R at d satisfies $C'(\lambda)^{wop}$.
② $\lambda > 0$ $d > \lambda/2$, R — doesnot — $C'(\lambda)^{wop}$

$\boxed{\text{Theorem:}}$ G a random group at density d.
① $d < \frac{1}{2}$ w.o.p. G is infinite hyperbolic torsion-free and dimension 2.
② $d > \frac{1}{2}$ w.o.p. G is either $\{1\}$ or $\mathbb{Z}_2$.

Recall the density model:
For $d \in [0, 1]$, $\ell \in \mathbb{N}$, pick $(2m-1)^{d\ell}$ reduced words of length ℓ independently uniformly.
Why not $d(2m-1)^{\ell}$?
"dimension"
dimension of a subspace = how many independent equations can you satisfy in the subspace.
Consider $(2m-1)^{\ell}$ words of length ℓ.
and $\{1^{st}\text{ letter} = a\}$, $\{2^{nd}\text{ letter} = a\}$, $\dots$, $\{\ell^{th}\text{ letter} = a\}$
$(2m-1)^{d\ell}$

Simple counting argument this

this set has a word containing $a^{d\ell}$ as a subword

[illegible] words cannot say $a = 1$

$\mathbb{F}_2 \rightsquigarrow$ even words cannot say ...

<u>Ball variant</u>: validity of all random group theorems seems to be preserved.

<u>Do we allow choosing the same word twice?</u> : doesn't matter.

<u>Floor or ceiling?</u> : doesn't matter.

<u>Reduced or cyclically reduced?</u> : doesn't matter.

Recall: d-density model ($d \in [0,1]$)
$\hookrightarrow$ choose $(2m-1)^{d\ell}$ relators of length $(\leq)\ \ell$.

<u>Today</u>: "$d=0$". $\rightsquigarrow$ choose k relators of length $(\leq)\ \ell$.

"few relator model" $\rightsquigarrow$ almost free.

<u>Proposition</u>: A random few relator group exponentially generically satisfies $C'(\frac{1}{6})$ and is hyperbolic.

<u>Proof</u>: if $\vec{g}$ is a tuple and $\vec{h} \subseteq \vec{g}$ and $\vec{g}$ satisfies $C'(\frac{1}{6})$, so does $\vec{h}$.

$0 < d < \frac{1}{2}$ satisfies $C'(\frac{1}{6})$ generically.

'Anything satisfied at $d < \ldots$ is satisfied by k-relator'

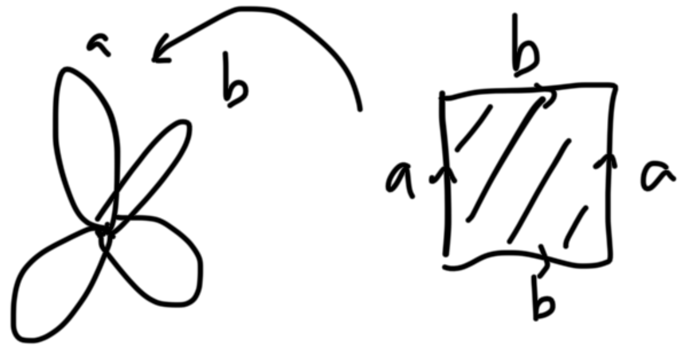
Theorem: With overwhelming probability in the few relator model, $G = \langle a_1, \dots, a_m \mid r_1, \dots, r_n \rangle$
Any subgroup of G generated by $m-1$ elements is free.

Motivating example:

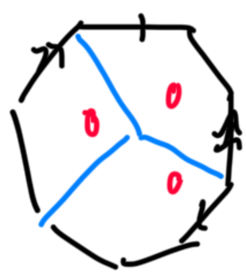
compact, closed oriented genus g

Any infinite index subgroup of $\pi_1(S)$ is free.

Proof: Claim: $\pi_1(S)$ non-compact is free.



$S \cong$ punctured $\underbrace{\text{closed surface}}$.



Sketch of proof of Theorem:

Find a class P of k-relator presentations that is 'generic' in the k-relator model, and is nice enough to have these properties.

$P_{\lambda,\mu} = \langle a_1, \dots, a_m \mid \vec{u} \rangle$ such that $\vec{u}$ has $C'(\lambda)$,

$\vec{u}$ does not contain a proper power,

every prefix w of u_i is such that:

there is not $H \le F_m$ ... Stallings graph has at most $\mu|w|$ edges,

$wh \in H$ is reduced for some $h \in H$,
$\mathrm{rank}(H) \leq r-1$ or $\mathrm{rank}(H) \leq r$ and $H \leq_{f.i.} F_m$.

If $\lambda \leq \frac{\mu}{15r+3\mu} \leq \frac{1}{6}$ then P is exponentially generic. And P satisfies all the theorems.

Theorems: × ~~For random few relator groups,~~ G a random few relator group, no finite-index subgroups are free.

× Any two m-tuples of elements generating a non-free subgroup are Nielsen-equivalent.

↳ Any automorphism of G lifts to an automorphism of F_m.

'almost free-ness':

Let G be a few-relator group, $L \geq 1$ an integer.

- Any subgroup rank $\leq L$ and infinite index is free
- Any subgroup rank $\leq L$ is quasiconvex
- Any non-trivial normal subgroup rank $\leq L$ is finite index
- $H_1, H_2 \leq G$ rank $\leq L$. $H_1 \cap H_2$ is f.g. and quasiconvex

$\rightsquigarrow$ not clear if all infinite-index subgroups are free.

Theorem: G random few-relator group
let $H = \langle h_1, \dots, h_k \rangle$ be an infinite order subgroup of F_m.
Then w.o.p. the map $F_m \to G$ is injective on H.

1-relator groups

Notation: $G_u = \langle A \mid u \rangle$.

Fact: $u, v \in F(A)$,
$\langle\langle u \rangle\rangle = \langle\langle v \rangle\rangle \iff u$ conjugate to v or v^{-1}.

Consequence: $G_u \cong G_v \iff \exists \varphi \in \mathrm{Aut}(F_A)$
$\varphi(u) \in \{v, v^{-1}\}$.

in particular, this solves the isomorphism problem for 1-relator groups.

$\mathbb{P}_n$: choose a random word of length $\leq n$ in $F(A)$ to be the relator

$\mathbb{Q}_n$: choose an isomorphism class of 1-relator groups at random st $u \leq n$

T: all isomorphism classes of 1-relator ~~su~~ groups.

Proposition: Let $X \subseteq T$ be a property of 1-relator groups. Let $Y = \{u \in F(A) \mid |u| \leq n, G_u \in X\}$ if $\mathbb{P}_n(Y) = o(\frac{1}{n})$ (or $o(e^{-n})$) then X is negligible (or exponentially negligible).

Proof: ??

Corollary: Exponentially generically, 1-relator groups are hyperbolic, every automorphism is induced by an automorphism of $F(A)$, every l-generated subgroup is free and quasi-convex if $l < |A|$.

$\boxed{\text{Theorem:}}$ $|A| \geq 3$, a 1-relator group is generically ~~residually~~ residually p-finite, and coherent.

Coherence: every f.g. subgroup is f.p.