05.02.2026 RG2

Critical densities for various properties

Extra ref: Y. Ollivier, Some small cancellation properties of random groups, 2007.

Recall: Thm: G Random group at density d, $0 \leq d \leq 1$.
- If $d < 1/2$, then w.o.p. G is infinite hyperbolic, torsion-free, geometric dimension 2.
- If $d > 1/2$, w.o.p. G is $\mathbb{Z}/2\mathbb{Z}$ or trivial.

Schedule:
1. van Kampen diagrams
2. Greendlinger's property + Dehn algo.
3. Proof!
4. Kazhdan's property (T).

1. van Kampen diagrams

Q: For $x, y \in F(A)$, when $x \underset{G}{=} y$?

$$\Leftrightarrow \quad xy^{-1} \underset{G}{=} 1_G. \qquad \text{(Word Problem!)}$$

Fact: A reduced word $w \in F(A)$ is equal 1_G iff it can be written as

$$\prod_{i=1}^{N} u_i r_i^{\pm 1} u_i \qquad u_i \in F(A), \quad r_i \in \vec{R}.$$

Pf:

$\Leftarrow$ Everything reduces to 1_G.

$\Rightarrow$: $\pi : F(A) \to G$. If $w \underset{G}{=} e$, then $w \in \ker(\pi)$.

$G = F(A)/\langle\langle \vec{R} \rangle\rangle$. Hence, $w \in \langle\langle \vec{R} \rangle\rangle$ and $\ker(\pi) = \langle\langle \vec{R} \rangle\rangle$. $\square$

Def. [van Kampen diagram]. A VKD over a presentation $P = \langle A, \vec{h} \rangle$ is a combinatorial map $\Delta \to Cay^2(P)$, where Δ is a connected, simply conn, planar 2-complex s.t.

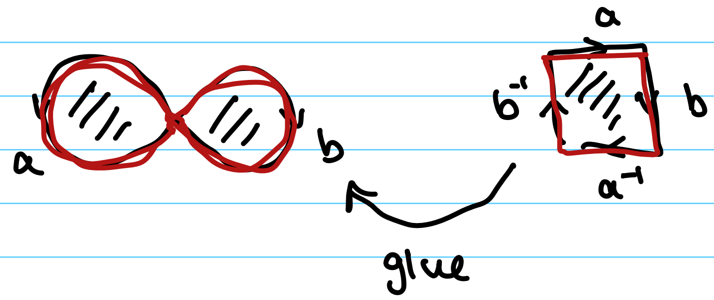
For every 2-cell in Δ, the label of the boundary cycle is a reduced word in $F(A)$ belongs to $(\vec{h})_*$.

$(\vec{h})_* = \{\vec{h}, (\vec{h})^{-1}, \text{ cyclic permutations for both}\}$.

E.g. $P = \langle a, b \mid \overbrace{aba^{-1}b^{-1}} \rangle$

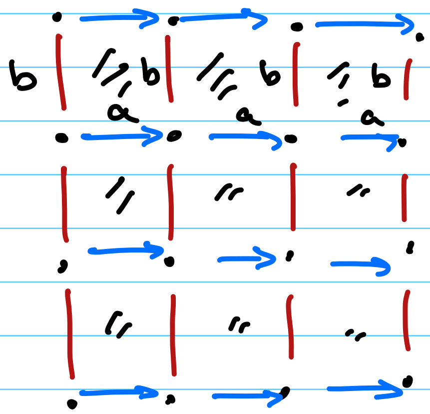
$Cay^2(P) = Pres^2(P)$

$Pres^2(P)$:



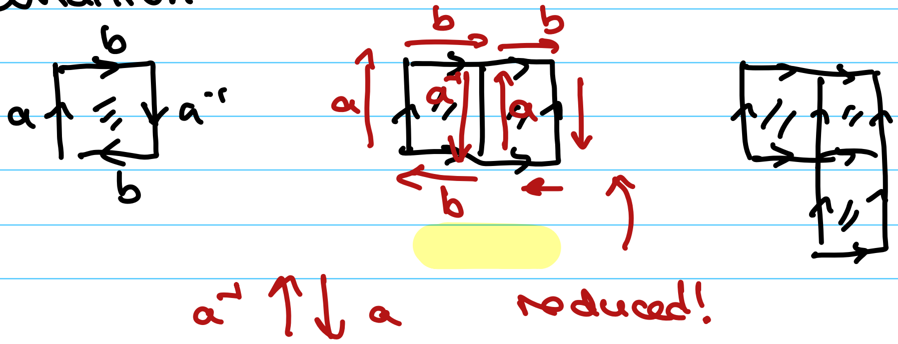
$Pres^2(P) = \mathbb{T}^2$.

$Cay^2(P)$:



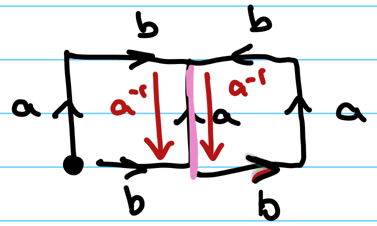
The VKD are combinatorial maps w/ boundary condition.

E.g.



Def: A VKD is reduced if no two cells meet in edge where, read off with orientation, the common edge has the same label.

E.g.

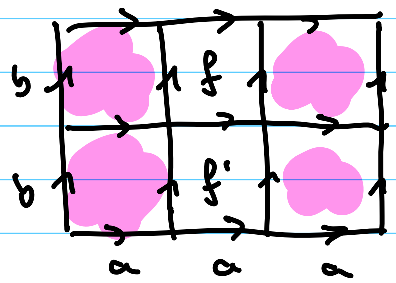


Fact: $C'(\lambda)$ s.c. property for a finite presentation $\langle A, \vec{r} \rangle$
<=> In a RVKD, a segment of consecutive edges in the boundary between adj faces f, f' has length $\leq \lambda \min(|\partial f|, |\partial f'|)$.

② Greendlinger's property + Dehn's algorithm

Def: [Greendlinger's property] G.P. holds for a finite pres. $\langle A, \vec{r} \rangle$ iff. in any RVKD D of $\langle A, \vec{r} \rangle$, $\exists$ faces f, f' for which we have segments of consecutive edges of $\partial f \cap \partial D$ / $\partial f' \cap \partial D$ of length at least $\frac{1}{2}|\partial f|$ / $\frac{1}{2}|\partial f'|$.

eg: $\langle a, b \mid [a,b] \rangle$



$$|\partial f \cap \partial D| = |\partial f' \cap \partial D| = \tfrac{1}{4}|\partial f| = \tfrac{1}{4}|\partial f'|.$$

But corner faces work!

Thm: Let $\langle A, \vec{r} \rangle$ be a fin. group presentation s.t. $C'(1/6)$ holds. Then Greenlinger's holds!

Cor: If $G = \langle A, \vec{r} \rangle$, G hyp, $\langle A, \vec{r} \rangle$ finite then Greendlinger's holds.

Def. [Dehn's algorithm].

① Fix a finite pres. $G = \langle A, \vec{R} \rangle$.

② Start with a reduced word $w \in F(A)$.

③ • If w contains more than half a relator:

$$w = w_1 u w_2, \qquad uv = 1 \quad (uv \text{ relator})$$
$$|v| < |u| \quad (\text{more than half})$$

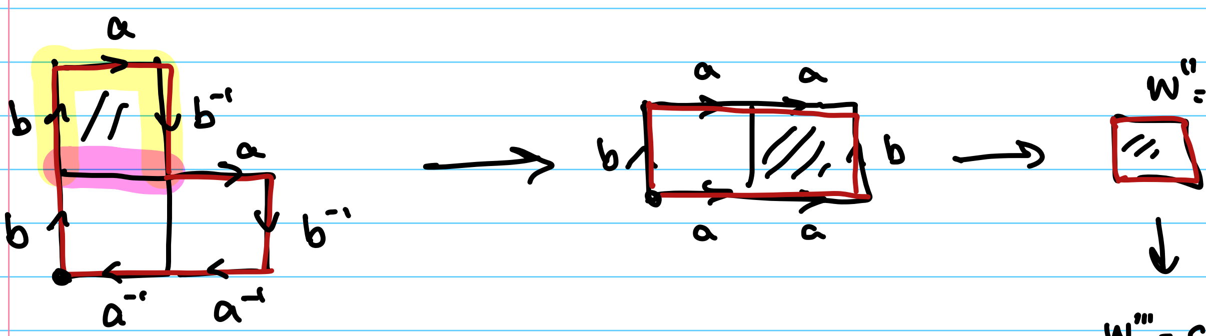
$\Rightarrow w' = w_1 v^{-1} w_2$ and then reduce w'.

• If not, then terminate. Output $w^{(n)}$. (previous w)

④ Repeat until you terminate.

Fact: Always terminates! If it terminates with empty word ε, then $w \underset{G}{=} 1$.

E.g.



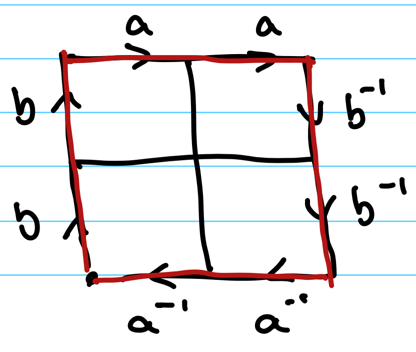
$w'' = bab^{-1}a^{-1}$.

$w''' = \varepsilon$.

$w = bbab^{-1}ab^{-1}a^{-1}a^{-1}$. $\qquad w' = baab^{-1}a^{-1}a^{-1}$

Non-eg. $w_n = a^n b^n a^{-n} b^{-n}$. $(n \geq 2)$

$n = 2$.



$G = \langle a, b \mid [a, b] \rangle$.

Def: [Dehn presentation] A fin. group presentation is a Dehn presentation if every trivial reduced word contains more than half a relator.

Prop. $\langle A, \vec{r} \rangle$ has Greendlinger's $\Rightarrow$ $\langle A, \vec{r} \rangle$ is a Dehn pres.

Cor. G hyp, f.p. $\Rightarrow$ G has a Dehn presentation $\langle A, \vec{r} \rangle$.

Thm: Let $0 \leq d \leq 1$.

- If $d < \frac{1}{5}$, a random finite presentation is Dehn and has Greendlinger's w.o.p.
- If $d > \frac{1}{5}$, a RFP. fails both properties. w.o.p.

③ Proof of Greendlinger's phase transition thm.

Pf (partial!)

Lem: For any $\varepsilon > 0$, w.o.p. we have:

Let D be RVKD s.t. $|D| \geq 2$. Then, there exist two faces f, f' of D s.t.

$$|\partial f \cap \partial D| \geq \ell \left(1 - \frac{5d}{2} - \varepsilon\right)$$

$$|\partial f' \cap \partial D| \geq \ell \left(1 - \frac{5d}{2} - \varepsilon\right)$$

Note: When $d < \frac{1}{5}$, $\ell \cdot \left(1 - \frac{5d}{2} - \varepsilon\right) > \frac{\ell}{2}$ (for small ε).

This shows $d < 1/5 \Rightarrow$ Greendlinger $\Rightarrow$ Dehn.

Pf: • Let D be RVKD, $|D| \geq 2$.

• Let f be a face which has the greatest number of edges on ∂D.

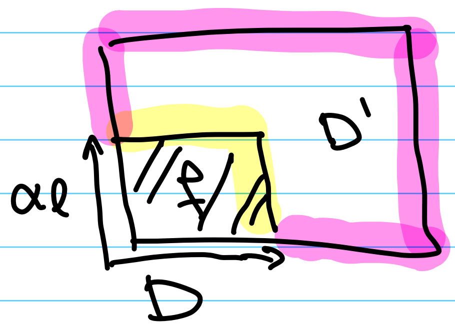
Say $|\partial f \cap \partial D| = \alpha \ell$.

• Suppose, that any other face has no more than $\beta \ell$ edges on ∂D.

• It suffices to show that $\beta \geq 1 - \frac{5d}{2} - \varepsilon$.

Suppose $\beta < 1 - \frac{5d}{2} - \varepsilon$.

Consider diagram D' which we obtain by removing f from D.



$$|\partial D'| = |\partial D| - \alpha\ell + (\ell - \alpha\ell)$$
$$= |\partial D| + \ell - 2\alpha\ell.$$

By def α, β,

$$|\partial D| \leq \underline{\beta\ell(|D| - 1) + \alpha\ell} \qquad (**)$$

$$\Rightarrow |\partial D'| \leq \beta\ell(|D| - 1) + \ell - \alpha\ell$$

Claim 1: For every $\varepsilon > 0$, w.o.p. every RVKD in a random group at density d satisfies

$$|\partial D| \geq (1 - 2d - \varepsilon) \cdot |D| \cdot \ell. \qquad //$$

Hence, assuming claim 1 we have:

$$\geq |\partial D| \geq \left(1 - 2d - \frac{\varepsilon}{2}\right)|D| \cdot \cancel{\ell}$$

$$|\partial D'| \geq \left(1 - 2d - \frac{\varepsilon}{2}\right)|D'| \cdot \ell = \left(1 - 2d - \frac{\varepsilon}{2}\right)\ell \cdot (|D| - 1).$$

Combining:

$$\left(1 - 2d - \frac{\varepsilon}{2}\right)|D| \leq \beta(|D| - 1) + \alpha$$

$$\left(1 - 2d - \frac{\varepsilon}{2}\right)(|D| - 1) \leq \beta(|D| - 1) + 1 - \alpha.$$

We assumed $\beta < 1 - \frac{5d}{2} - \varepsilon$.

Hence,

$$|D| < \frac{\alpha + \frac{5d}{2} - 1 + \varepsilon}{\frac{d}{2} + \frac{\varepsilon}{2}} \qquad (1)$$

$$|D| < \frac{d/2 + 1 - \alpha + \varepsilon/2}{\frac{d}{2} + \frac{\varepsilon}{2}} \qquad (2)$$

Claim2: This implies $|D| < 3$.

If $\alpha \leq 1 - d - \frac{\varepsilon}{4}$, (2) gives $|D| < \dfrac{\frac{3d}{2} + \frac{3\varepsilon}{4}}{\frac{d}{2} + \frac{\varepsilon}{2}} < 3$.

$\alpha > 1 - d - \varepsilon/4$, (1) gives $|D| < 3$.

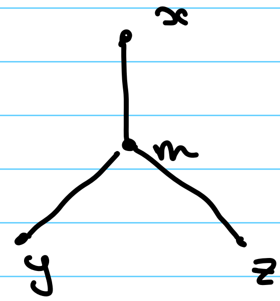
As $|D| \geq 2$, then $|D| = 2$.

$|D| = 2$ is dealt with by Ollivier. $\square$

④ Kazhdan's property T.

Def. A median space (X, d) is a metric space s.t. $\forall\, x, y, z \in X$, $\exists!\ m = m(x,y,z)$ s.t.

1. $d(x,m) + d(m,y) = d(x,y)$
2. $d(z,m) + d(m,y) = d(z,y)$
3. $d(x,m) + d(m,z) = d(x,z)$



Def. A group G has Kazhdan's property (T) if any isometric action of G on a median space has bounded orbit.

Intuition: groups w/ (T) don't have splittings!
(free prod. + HNN)

Thm: Let $0 \le d \le 1$. If $d < \frac{1}{3}$, then at density d a RFP does not have (T) w.o.p.
If $d > 1/3$, at density d ... has (T) w.o.p.