K-Theory lecture 4

The Functor K_0

Last time: functor K_0 from the category of unital C^*-alg.s to the category of abelian groups.
This time: extend K_0 to all C^*-alg.s

DEF

A non-unital C^*-alg.. We have the split exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

where $\pi(a+\alpha 1_{\tilde{A}}) = \alpha$ and $\lambda(\alpha) = \alpha 1_{\tilde{A}}$
Define $\underline{K_0(A)}$ to be the kernel of the homomorphism

$$K_0(\pi) : K_0(\tilde{A}) \to K_0(\mathbb{C})$$

($\tilde{A}$ and $\mathbb{C}$ are unital so this makes sense).

- $K_0(A)$ is an abelian group as it is a subgroup of $K_0(\tilde{A})$.

- For $p \in P_\infty(A)$ we have
  $K_0(\pi)([p]_0) = [\pi(p)]_0 = 0$
  So $[p]_0 \in \ker K_0(\pi) = K_0(A)$, and we get the map
  $[-]_0 : P_\infty(A) \to K_0(A)$.

- We have a S.E.S.
  $$0 \to K_0(A) \xrightarrow{\star} K_0(\tilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \to 0$$

where $\star$ is $K_0(\iota)$ if A is unital, and the inclusion otherwise. If A is unital, by exactness,

$$\ker(K_0(\pi)) = \operatorname{im}(K_0(\iota)) \cong K_0(A) \leftarrow \text{old definition}.$$

Let $\varphi : A \to B$ be a $*$-homom.. Then

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "\varphi"'] & \tilde{A} \arrow[r, "\pi_A"] \arrow[d, "\tilde{\varphi}"] & \mathbb{C} \arrow[d, equal] \\ B \arrow[r, "\iota_B"'] & \tilde{B} \arrow[r, "\pi_B"'] & \mathbb{C} \end{tikzcd}$$

Applying the functor K_0 for unital C^*-alg.s, we get

By the previous S.E.S.

$$\begin{tikzcd} K_0(A) \arrow[r] \arrow[d, dashed, "\exists! K_0(\varphi)"'] & K_0(\tilde{A}) \arrow[r, "K_0(\pi_A)"] \arrow[d, "K_0(\tilde{\varphi})"] & K_0(\mathbb{C}) \arrow[d, equal] \\ K_0(B) \arrow[r] & K_0(\tilde{B}) \arrow[r, "K_0(\pi_B)"'] & K_0(\mathbb{C}) \end{tikzcd}$$

- $K_0(\varphi)$ is the restriction of $K_0(\tilde{\varphi})$ to $K_0(A)$.
- If A, B unital, $K_0(\varphi)$ agrees with the previous definition: the diagram still commutes using this $K_0(\varphi)$.
- $K_0(\varphi)([p]_0) = [\varphi(p)]_0 \qquad p \in P_\infty(A)$ whether A is unital or not.

PROP 4.1.3

(i) $K_0(id_A) = id_{K_0(A)} \quad \forall A$

(ii) $\varphi : A \to B$, $\psi : B \to C$ $*$-homom.s then $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$

} K_0 is a functor

(iii) $K_0(0) = 0$

(iv) $K_0(0_{B,A}) = 0_{K_0(B), K_0(A)}$

Pf.

(i),(ii) by 3.2.4 and $\widetilde{(id_A)} = id_{\tilde{A}}$, $\widetilde{(\psi \circ \varphi)} = \tilde{\psi} \circ \tilde{\varphi}$

(iii) $\tilde{0} = \mathbb{C}$. If $A = 0$ we get

$$0 \to 0 \to \mathbb{C} \xrightarrow{\pi} \mathbb{C} \to 0$$

with $\pi = id_{\mathbb{C}}$.

$K_0(\{0\}) = \ker(K_0(\pi)) = 0$

(iv) Same proof as 3.2.4 (follows from (iii))

PROP 4.1.4

(i) If $\varphi, \psi : A \to B$ are homotopic $*$-homom.s then
$$K_0(\varphi) = K_0(\psi),$$

(ii) If A and B are homotopy equivalent then $K_0(A) \cong K_0(B)$,
If $\varphi : A \to B$, $\psi : B \to A$ is a homotopy then
$$K_0(\varphi) : K_0(A) \to K_0(B) \quad \text{and} \quad K_0(\psi) : K_0(B) \to K_0(A)$$
are isomorphisms with $K_0(\varphi)^{-1} = K_0(\psi)$

Pf

(i) $\varphi \sim_h \psi \Rightarrow \tilde{\varphi} \sim_h \tilde{\psi} \Rightarrow K_0(\tilde{\varphi}) = K_0(\tilde{\psi})$
$\Rightarrow K_0(\varphi) = K_0(\psi)$ (restrictions to $K_0(A)$)

(ii) Follows from (i). ∎

Example

Cone of A is
$$CA = \{ f \in C([0,1], A) : f(0) = 0 \}.$$

Suspension
$$SA = \{ f \in C([0,1], A) : f(0) = f(1) = 0 \}.$$

There is a S.E.S.
$$0 \to SA \xrightarrow{\iota} CA \xrightarrow{\pi} A \to 0$$
where ι is inclusion and $\pi(f) = f(1)$.

CA is homotopy equivalent to 0:
$\varphi_t : CA \to CA$
$\varphi_t(f)(s) = f(st)$.

The map $t \mapsto \varphi_t(f)$ is cts. $\forall f \in CA$, $\varphi_0 = 0$ and $\varphi_1 = id$. So
$0 : CA \to 0$, $0 : 0 \to CA$ is a homotopy.

Hence $K_0(CA) = 0$.

Last time: A unital, then

$$K_0(A) = \{ [p]_0 - [q]_0 : p, q \in P_\infty(A) \}$$
$$= \{ [p]_0 - [q] : p, q \in P_n(A), n \in \mathbb{N} \}$$

What if A non-unital?

The Scalar Mapping

$$0 \to A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

Define the <u>scalar mapping</u> s to be

$$s = \lambda \circ \pi : \tilde{A} \to \tilde{A}$$

i.e. $s(a + \alpha 1) = \alpha 1$

- $\pi(s(x)) = \pi(x)$
- $x - s(x) \in A \quad \forall x.$

Let $s_n : M_n(\tilde{A}) \to M_n(\tilde{A})$ be the $*$-homom. induced by s (componentwise s). The image of s_n is $M_n(\mathbb{C})$ and $x - s_n(x) \in M_n(A)$. Write s instead of s_n.

$x \in \tilde{A}$ or $M_n(\tilde{A})$ is a <u>scalar element</u> if $x = s(x)$.

$x \in M_n(\tilde{A})$ scalar if all its entries are scalar multiples of $1_{\tilde{A}}$.

The scalar mapping is natural: if $\varphi : A \to B$ a $*$-homom. then

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{s} & \tilde{A} \\ \tilde{\varphi}\downarrow & \circlearrowright & \downarrow\tilde{\varphi} \\ \tilde{B} & \xrightarrow{s} & \tilde{B} \end{array}.$$

<u>PROP. 4.2.2</u>

For any C^*-alg. A,

$$K_0(A) = \{ [p]_0 - [s(p)]_0 : p \in P_\infty(\tilde{A}) \}.$$

Moreover,

(i) For $p, q \in P_\infty(\tilde{A})$, T.F.A.E. :

(a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$

(b) $\exists\, k, l \in \mathbb{N}$ s.t. $p \oplus 1_k \sim_0 q \oplus 1_l$ in $P_\infty(\tilde{A})$

(c) $\exists$ scalar projections r_1 and r_2 s.t. $p \oplus r_1 \sim_0 q \oplus r_2$.

(ii) If $[p]_0 - [s(p)]_0 = 0$ then $\exists\, m \in \mathbb{N}$ with

$$p \oplus 1_m \sim s(p) \oplus 1_m.$$

(iii) If $\varphi : A \to B$ a $*$-homom. then

$$K_0(\varphi)\big([p]_0 - [s(p)]_0\big) = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0 \quad \forall p \in P_\infty(\tilde{A}).$$

Pf

For $p \in P_\infty(\tilde{A})$,

$$K_0(\pi)([p]_0 - [s(p)]_0) = [\pi(p)]_0 - [(\pi \circ s)(p)] \underset{\substack{\uparrow \\ \pi = \pi \circ s}}{=} 0$$

$\Rightarrow [p]_0 - [s(p)]_0 \in \ker(K_0(\pi)) = K_0(A)$.

Conversely let $g \in K_0(A)$. $\exists\, n \in \mathbb{N}$, projections $e, f \in M_n(\tilde{A})$ s.t.

$g = [e]_0 - [f]_0$. Let

$$p = \begin{pmatrix} e & 0 \\ 0 & 1_n - f \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix}.$$

Then $p, q \in P_{2n}(\tilde{A})$ and

$[p]_0 - [q]_0 = [e]_0 + [1_n - f]_0 - [1_n]_0 = [e]_0 - [f]_0 = g$,

$q = s(q)$ and $K_0(\pi)(g) = 0$, so

$[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = K_0(s)(g) = (K_0(\lambda) \circ K_0(\pi))(g) = 0$

$\Rightarrow g = [p]_0 - [s(p)]_0 + [s(p)]_0 - [q]_0 = [p]_0 - [s(p)]_0$.

(i) (a) $\Rightarrow$ (c)

Let $p, q \in P_\infty(\tilde{A})$ with $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$. Then $[p \oplus s(q)]_0 = [q \oplus s(p)]_0$ and so $p \oplus s(q) \sim_s q \oplus s(p)$ in $P_\infty(\tilde{A})$ (prop. from last week). Also from last week $\exists\, n \in \mathbb{N}$ s.t.

$$p \oplus \underbrace{s(q) \oplus 1_n}_{r_1} \sim_0 q \oplus \underbrace{s(p) \oplus 1_n}_{r_2}.$$

(c) $\Rightarrow$ (b)

r_1, r_2 scalar projections of dim. k, ℓ resp., then

$r_1 \sim_0 1_k$ and $r_2 \sim_0 1_\ell$ (exercise)

So $p \oplus 1_k \sim_0 q \oplus 1_\ell$.

(b) $\Rightarrow$ (a)

$[p \oplus 1_k]_0 - [s(p+1_k)]_0 = [p]_0 - [s(p)]_0$, so it suffices to show that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ when $p \sim_0 q$.

Suppose $p = v^*v$, $q = vv^*$ for some partial isometry $v \in M_{m,n}(\tilde{A})$.

$s(v) \in M_{m,n}(\mathbb{C}) \subseteq M_{m,n}(\tilde{A})$ ($s(v)$ componentwise).

Then $s(v)^* s(v) = s(p)$ and $s(v) s(v)^* = s(q)$, so $s(p) \sim_0 s(q)$.

Hence $[p]_0 = [q]_0$ and $[s(p)]_0 = [s(q)]_0$, so (a) holds.

(ii) If $[p]_0 - [s(p)]_0 = 0$, then $p \sim_s s(p)$ (last week).

$\Rightarrow \exists\, m \in \mathbb{N}$ s.t. $p \oplus 1_m \sim s(p) \oplus 1_m$.

(iii) $K_0(\varphi)([p]_0 - [s(p)]_0) = K_0(\tilde{\varphi})([p]_0 - [s(p)]_0)$

$= [\tilde{\varphi}(p)]_0 - [\tilde{\varphi}(s(p))]_0 = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$ ∎

LEM 4.2.3

$\varphi : A \to B$ a $*$-hom. Suppose $g \in K_0(A)$ satisfies $g \in \ker(K_0(\varphi))$.
Then

(i) $\exists\, n \in \mathbb{N}$, $p \in P_n(\tilde{A})$, and unitary $u \in M_n(\tilde{B})$ s.t.

$$g = [p]_0 - [s(p)]_0 \quad \text{and}$$
$$u \tilde{\varphi}(p) u^* = s(\tilde{\varphi}(p))$$

(ii) If φ is surjective, $\exists\, p \in P_\infty(\tilde{A})$ s.t.

$g = [p]_0 - [s(p)]_0$ and $\tilde{\varphi}(p) = s(\tilde{\varphi}(p))$.

Pf

By 4.2.2 $\exists\, k \in \mathbb{N}$, $p_1 \in P_k(\tilde{A})$ with $g = [p_1]_0 - [s(p_1)]_0$ and $[\tilde{\varphi}(p_1)]_0 - [s(\tilde{\varphi}(p_1))]_0 = 0$. Also

$\tilde{\varphi}(p_1) \oplus 1_m \sim s(\tilde{\varphi}(p_1)) \oplus 1_m$ for some $m \in \mathbb{N}$.

Let $p_2 = p_1 \oplus 1_m$. Then $p_2 \in P_{k+m}(\tilde{A})$ and

$g = [p_2]_0 - [s(p_2)]_0$, and

$\tilde{\varphi}(p_2) = \tilde{\varphi}(p_1) \oplus 1_m \sim s(\tilde{\varphi}(p_1)) \oplus 1_m = s(\tilde{\varphi}(p_2))$.

Let $n = 2(k+m)$ and 0_{k+m} be the 0-projection in $M_{k+m}(\tilde{A})$.

Set $p = p_2 \oplus 0_{k+m}$, so $p \in P_n(\tilde{A})$.

$g = [p]_0 - [s(p)]_0$ and

$u\tilde{\varphi}(p)u^* = s(\tilde{\varphi}(p))$ for some unitary $u \in M_n(\tilde{B})$ (2.2.8)

(ii) Take n, p, and u from (i) and ~~set $p_1 = p$~~ rename p to p_1. By LEM 2.1.7 $\exists$ unitary $v \in M_{2n}(\tilde{A})$ with $\tilde{\varphi}(v) = \operatorname{diag}(u, u^*)$. Let $p = v \operatorname{diag}(p_1, 0_n) v^*$. Then p is a projection in $M_{2n}(\tilde{A})$ and

$$\tilde{\varphi}(p) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}\begin{pmatrix} \tilde{\varphi}(p_1) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} s(\tilde{\varphi}(p_1)) & 0 \\ 0 & 0 \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

So $s(\tilde{\varphi}(p)) = \tilde{\varphi}(p)$, and

$g = [p]_0 - [s(p)]_0$ since $p \sim_0 p_1$. $\blacksquare$

Half + Split Exactness and Stability

LEM 4.3.1

Let
$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$ be a S.E.S. of C^*-alg.s, and $n \in \mathbb{N}$. Then

(i) $\tilde{\varphi}_n : M_n(\tilde{I}) \to M_n(\tilde{A})$ is injective

(ii) For $a \in M_n(\tilde{A})$,

$$a \in \operatorname{im}(\tilde{\varphi}_n) \iff \tilde{\psi}_n(a) = s_n(\tilde{\psi}_n(a)).$$

PROP 4.3.2 'Half-exactness'

Every S.E.S. of C^*-alg.s
$$0 \to I \xrightarrow{\varphi} A \xrightarrow{\psi} B \to 0$$ induces an exact seq. of abelian gp.s
$$K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B).$$

Pf

By functoriality of K_0,

$$K_0(\psi)\circ K_0(\varphi) = K_0(\psi\circ\varphi) = K_0(0) = 0.$$

$\Rightarrow \mathrm{im}(K_0(\varphi)) \subseteq \ker(K_0(\psi))$.

Conversely, let $g \in \ker(K_0(\psi))$. By ~~the~~ lemma 4.2.3, $\exists\, n\in\mathbb{N}$, $p \in P_n(\tilde{A})$ s.t. $g = [p]_0 - [s(p)]_0$ and $\tilde\psi(p) = s(\tilde\psi(p))$.

By ~~the~~ lemma 4.3.1, $\exists\, e \in M_n(\tilde{I})$ with $\tilde\varphi(e) = p$, and $\tilde\varphi$ is injective. So e is a projection, and

$$g = [\tilde\varphi(e)]_0 - [s(\tilde\varphi(e))]_0 = K_0(\varphi)([e]_0 - [s(e)]_0) \in \mathrm{im}(K_0(\varphi))$$ ∎

PROP. 4.3.3 (Split-exactness)

Every split exact sequence of C^*-alg.s

$$0 \to I \xrightarrow{\varphi} A \underset{\lambda}{\overset{\psi}{\rightleftarrows}} B \to 0$$

induces a split exact sequence of abelian groups

$$0 \to K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \underset{K_0(\lambda)}{\overset{K_0(\psi)}{\rightleftarrows}} K_0(B) \to 0.$$

Pf

By PROP. 4.3.2, the sequence is exact at $K_0(A)$. By functoriality,

$$\mathrm{id}_{K_0(B)} = K_0(\mathrm{id}_B) = K_0(\psi)\circ K_0(\lambda).$$

So exact at $K_0(B)$.

W.T.S. $K_0(\varphi)$ is injective.

Let $g \in \ker(K_0(\varphi))$. By 4.2.3(i), $\exists\, n\in\mathbb{N}$, projection $p \in P_n(\tilde{I})$ and unitary $u \in M_n(\tilde{A})$ with

$$g = [p]_0 - [s(p)]_0 \quad \text{and} \quad u\tilde\varphi(p)u^* = s(\tilde\varphi(p)).$$

Let $v = (\tilde\lambda\circ\tilde\psi)(u^*)u$. Then v is a unitary in $M_n(\tilde{A})$ and $\tilde\psi(v) = 1$.

By 4.3.1(ii) $\exists\, w \in M_n(\tilde{I})$ with $\tilde\varphi(w) = v$.

$\tilde\varphi$ is injective, so w is unitary.

$$\tilde\varphi(wpw^*) = v\tilde\varphi(p)v^* = (\tilde\lambda\circ\tilde\psi)(u^*)\, s(\tilde\varphi(p))\,(\tilde\lambda\circ\tilde\psi)(u)$$
$$= (\tilde\lambda\circ\tilde\psi)(u^* s(\tilde\varphi(p)) u) = (\tilde\lambda\circ\tilde\psi)(\tilde\varphi(p)) = s(\tilde\varphi(p)) = \tilde\varphi(s(p))$$

By injectivity of $\tilde{\varphi}$, $upu^* = s(p)$.

$\Rightarrow p \sim_u s(p)$ in $M_n(\tilde{I})$

$\Rightarrow g = 0$.

Could also prove this using the L.E.S. in K-thy. from ch. 9.

PROP. 4.3.4

$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$.

If $\iota_A : A \to A \oplus B$, $\iota_B : B \to A \oplus B$ are the canonical inclusion mappings, then

$$K_0(\iota_A) \oplus K_0(\iota_B) : K_0(A) \oplus K_0(B) \to K_0(A \oplus B)$$

is an isomorphism.

Pf

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r, "\alpha"] \arrow[d, equal] & K_0(A) \oplus K_0(B) \arrow[r, "\beta"] \arrow[d, "K_0(\iota_A) \oplus K_0(\iota_B)"] & K_0(B) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r, "K_0(\iota_A)"'] & K_0(A \oplus B) \arrow[r, "K_0(\pi_B)"'] & K_0(B) \arrow[r] & 0
\end{tikzcd}$$

where $\alpha(g) = (g, 0)$, $\beta(g, h) = h$, $\pi_B(a, b) = b$

commutes: $\pi_B \circ \iota_A = 0$ and $\pi_B \circ \iota_B = id_B$,

So the middle arrow is an isomorphism.

Example

By split-exactness,

$$K_0(\tilde{A}) \cong K_0(A) \oplus \mathbb{Z} \quad \forall A.$$

K_0 is not exact:

Example From various examples/exercises:

$$0 \to C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \to 0$$

$K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^2$ and $K_0(C([0,1])) \cong \mathbb{Z}$.

So $K_0(\psi)$ isn't surjective.

Example

Let H be a separable infinite-dim. Hilbert space, K the ideal of cpct. operators in $B(H)$.
$\frac{B(H)}{K}$ is the Calkin algebra $Q(H)$. S.E.S.

$$0 \to K \xrightarrow{\iota} B(H) \xrightarrow{\pi} Q(H) \to 0.$$

$K_0(B(H)) \cong 0$ and $K_0(K) \cong \mathbb{Z}$

So K_0 isn't injective.

PROP 4.3.8 (Stability)

Let $n \in \mathbb{N}$. Then $K_0(A) \cong K_0(M_n(A))$.

The $*$-homom.

$$\lambda_{n,A} : A \to M_n(A)$$
$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

induces an isomorphism $K_0(\lambda_{n,A}) : K_0(A) \to K_0(M_n(A))$.

Pf (idea)

Formalise the fact that $P_\infty(A) = P_\infty(M_n(A))$.