A Crash Course on Boundaries

Page • Talia Shlomovich

This talk will be a survey talk on boundaries, where we compare and contrast 3 types of boundary: Gromov, Visual and Morse.

0. Motivating ideas

Roughly speaking, a boundary of a metric space X (group G), denoted ∂X, is a topological space which encodes the behaviour of X (or G) at infinity.

But what does this mean? What sort of behaviour are we interested in?

We can form a sensible list of properties (desiderata) that it would be useful for ∂X to satisfy:

- X, Y q.i. $\implies \partial X \cong \partial Y$ (comment: a nice consequence of this is that $\partial G \cong \partial X$, where $X = \mathrm{Cay}(G, S)$.)
- $X \cup \partial X$ is compact (coinciding with existing ideas of the boundary of a metric space)
- If $G \curvearrowright X$ nicely (properly, cocompactly, by isometries), then the action of G descends to an action on ∂X
- $\partial \mathbb{H}^2 \cong S^1, \partial \mathbb{E}^2 \cong S^1$

The vagueness of this definition allows for many different types of boundaries. Boundaries vary depending on which topological spaces they are defined on and which of the desiderata we want to be satisfied. There is usually a trade-off: boundaries defined on larger classes of spaces tend to satisfy fewer of the desiderata, and vice versa.

Example: Some types of boundaries you might have heard of are: Gromov boundary, Visual boundary, Morse boundary, Bowditch boundary, Floyd boundary, Martin boundary, $\mathbb{Z}$-boundary, ...

> **Question:** When, if at all, can we unify these boundaries?

> **Answer:** There is (currently) no single notion of infinity compatible with all spaces/groups and geometries. But, all these boundaries are equivalent when X

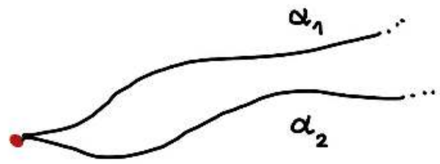
> is a hyperbolic space, and all the desiderata are satisfied. Hyperbolic spaces really are nice!

1. Gromov boundary

Def: Recall from the course that a space X is **hyperbolic** if triangles are thin, and a group G is **hyperbolic** if it acts properly and coboundedly on a hyperbolic metric space.

Def [Gromov '85]: The **Gromov boundary** of a δ-hyperbolic metric space (X, d), denoted ∂X, is the set of geodesic rays $\alpha : [0, \infty) \to X$ up to asymptotic equivalence of rays.

Two rays α_1, α_2 are asymptotically equivalent if there exists $K > 0$ s.t. for all $t > 0$, $d(\alpha_1(t), \alpha_2(t)) < K$.



For a point $p \in \partial X$, we write that $p = [\alpha]$ where α is a representative of the equivalence class.

We can also define the Gromov boundary based at a point, $\partial_{x_0} X$.

Examples:

$\partial \mathbb{Z} = \{x_1, x_2\}$

$\partial F_n, n \geq 2$ = Cantor set

$\partial \pi_1(S_g), g \geq 2 = S^1$

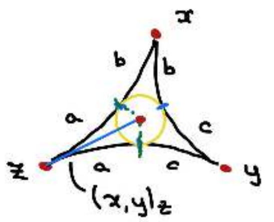
$\partial \mathbb{H}^2 = S^1$.

Lemma: If X is also proper, then equivalent rays are uniformly close with constant $K = 2\delta$.

Topology on the Gromov boundary

Def: For a triple $x, y, z \in X$, the Gromov product $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$.

Roughly, this is the distance that geodesics from z to x and z to y travel close to each other before diverging.

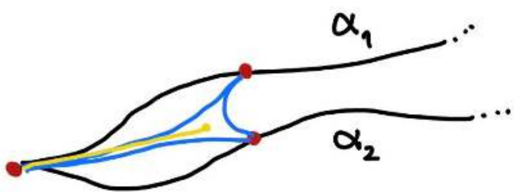


$$
\begin{aligned}
(x,y)_z &= \tfrac{1}{2}\left(d(x,z) + d(y,z) - d(x,y)\right)\\
&= \tfrac{1}{2}\left(a + b + a + c - (b + c)\right)\\
&= \tfrac{1}{2}(2a) = a
\end{aligned}
$$

Def: Let $x_0 \in X$ be a basepoint. Then, for any $p \in \partial_{x_0} X$ and $r > 0$, we define the basis of open sets

$$V(p,r) := \{q \in \partial_{x_0} X \mid \text{ for some geodesic rays } \alpha_1, \alpha_2 \text{ starting at } x_0, \text{s.t. } p = [\alpha_1], q = [\alpha_2] \text{ we have } \liminf (\alpha_1(t), \alpha_2(t))_{x_0} \geq r\}.$$

Thus $V(p,r)$ consists of equivalence classes of geodesic rays based at x_0 which stay 2δ-close to a geodesic ray representing p for approximately the distance r. We topologise the boundary ∂X by taking the $\{V(p,r) | r \geq 0\}$ as the open sets. Thus two geodesic rays starting at x_0 are "close at infinity" if they stay 2δ-close for a long time. This topology is known as the **Gromov topology**.



Prop: The cone topology is independent of the choice of basepoint x_0.

> **Question**: Why is the Gromov boundary only defined on hyperbolic spaces?

> **Answer**: In hyperbolic spaces, the Gromov product tells us how far geodesics fellow-travel. Large Gromov product = large distance that rays fellow travel. But if we have flats, then rays linearly diverge, but the Gromov product tends to infinity, so we lose the correspondence between the Gromov product and our fellow-travelling distance. The definition above breaks down.

We should also check well-definedness:

Thm [Gromov '85]: If X, Y are q.i., then $\partial X \cong \partial Y$.

2. Visual boundary

As a set, the visual boundary is defined very similarly to the Gromov boundary.

Def: CAT(0) space

Def [Cartan ~20s, Gromov '85]: The **visual boundary** of a complete $\mathrm{CAT}(0)$ space X is the set $\partial_{vis} X := \{[\alpha] \mid \sim\}$, endowed with the cone topology.

Def: The **cone topology** encapsulates the idea of fellow-travelling geodesics, however without the Gromov product. The basis of open sets is defined by:

$$U(\alpha_1, T, \varepsilon) := \{[\alpha_2] \mid \alpha_2 \text{ geodesic ray based at } x_0 \text{ and } \forall t < T, d(\alpha_1(t), \alpha_2(t)) < \varepsilon\}.$$

Examples:

$\partial_{vis} \mathbb{Z}^2 = S^1$

$\partial_{vis} \mathbb{Z}^n = S^{n-1}$

Lemma: The visual boundary is also independent of basepoint x_0.

Example 3.4 (Croke-Kleiner space). In [12], Croke and Kleiner showed that boundaries of CAT(0) spaces are not invariant under quasi-isometry. Their example involved the Salvetti complex S_Γ of a certain right-angled Artin group (RAAG). We briefly recall their construction. Let A_Γ denote the RAAG associated to the graph Γ in Figure 5. That is,

$$A_\Gamma = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle .$$

Thm: [Croke, Kleiner '98] In this group, the visual boundary of a CAT(0) group is not well-defined. Quasi-isometric CAT(0) spaces can have nonhomeomorphic boundaries!

3. Morse boundary

The Morse boundary encodes "hyperbolic directions" in a metric space. We will see what this means later.

Def: [Cordes '17] A geodesic α in a metric space is called N-**Morse**, where N is a function $N : [1, \infty) \times [0, \infty) \to [0, \infty)$, if for any (λ, ε)-quasi-geodesic σ with endpoints on α, we have $\sigma \subset \mathcal{N}_{N(\lambda,\varepsilon)}(\alpha)$, the $N(\lambda, \varepsilon)$-neighbourhood of α. We call the function $N(\lambda, \varepsilon)$ a **Morse gauge**. A geodesic is **Morse** if it is N-Morse for some gauge N.

Similar to before,

$$\partial_* X_{x_0} = \{[\alpha] \mid \alpha : [0, \infty) \to X \text{ Morse geodesic}, \alpha(0) = x_0\}$$

where equivalence is defined as previously.

We equip this set with a direct limit topology, with the direct limit taken over the Morse gauges. Firstly, we define **strata**:

$$\partial_*^N X_{x_0} = \{[\alpha] \mid \text{ there exists } \beta \in [\alpha] \text{ that is an } N\text{-Morse geodesic ray with } \beta(0) = x_0\}.$$

The strata have a similar topology to that of the visual boundary of hyperbolic spaces. We may consider a partial ordering on the set of all Morse gauges, and use that to define a direct limit. Specifically, we say $N \leq N'$ iff $N(\lambda, \varepsilon) \leq N'(\lambda, \varepsilon)$ for all λ, ε.

Denoting the set of all Morse gauges by $\mathcal{M}$, this is

$$\partial_* X_{x_0} = \lim_{\mathcal{M}} \partial_*^N X_{x_0}.$$

(This relies on showing that the inclusion $i : \partial_M^N X_p \hookrightarrow \partial_M^{N'} X_p$ is continuous.)

Examples:

$\partial_* \mathbb{Z}^n, n \geq 2 = \varnothing$

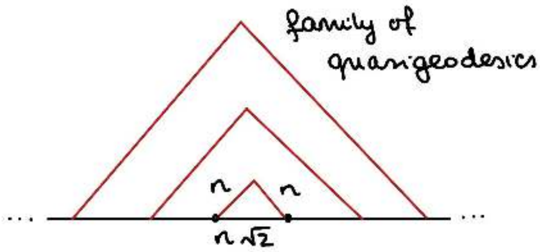
$\partial_*(\mathbb{Z} * \mathbb{Z}^2)$ = Cantor set

Lemma: Like before, the Morse boundary is shown to be independent of the basepoint x_0.

Lemma: (Morse Lemma) [Morse 1924] Let X be a hyperbolic space. Then there exists a Morse gauge $N : [1, \infty) \times [0, \infty) \to [0, \infty)$ such that all the geodesics of X are N-Morse.

It is a fact that this result holds in both directions. Therefore, this property (referred to as the Morse property in the literature) characterises hyperbolic spaces.

Example: No infinite geodesic in $\mathbb{R}^2$ has the Morse property.



4. Conclusions

When do these coincide?

Fact: When X is a (hyperbolic?) $\mathrm{CAT}(0)$ visibility space, $\partial X = \partial_{vis} X$. (Visibility space: for all $x, y \in \partial X$, there is a geodesic line which connects them.)

Fact: When X hyperbolic, $\partial X = \partial_* X$.

In summary,

	Gromov	*Visual*	*Morse*
Defining property of X	Hyperbolic	Complete CAT(0)	Proper, geodesic
Q.i. implies same boundary	✓	✕	✓
$X \cup \partial X$ compact	✓ (if X proper)	✓ (if X proper)	✕ (strata compact, but not boundary itself)
Actions descend	✓	✓	✓
Hyperbolic/Euclidean plane have boundary S^1	✓ (but not well-defined on E^2)	✓	✓ (but empty on E^2)