Deformation Theory Group Talk II

Deformation Theory II

Let $k = \bar{k}$ be a field $D = k[t]/\langle t^2 \rangle$ finite k-alg. with maximal ideal $m = \langle t \rangle$. As a scheme, $\operatorname{Spec} D$ consists of a unique point.

Fact. $T_x X \simeq \{ \varphi : D \to X,\ \varphi(0) = x \}$. [Idea: a D-point is a choice of a k-point of X and a tangent direction at x ("infinitesimal neighborhood").

§1. More on flatness

Recall. A ring, M an A-mod. There are functors $\operatorname{Tor}_i^A(M,-)$ s.t. for any s.e.s $0 \to N' \to N \to N'' \to 0$ there is a l.e.s

$$\cdots \to \operatorname{Tor}_2^A(M,N'') \to \operatorname{Tor}_1^A(M,N') \to \operatorname{Tor}_1^A(M,N)$$

$$\to \operatorname{Tor}_1^A(M,N'') \to \operatorname{Tor}_0^A(M,N') \to \operatorname{Tor}_0^A(M,N) \to \operatorname{Tor}_0^A(M,N'') \to 0$$

with $\operatorname{Tor}_0^A(M,N) \simeq M \otimes_A N$.

Remark. Note that

M is flat over A $\iff$ $\forall\, N$ A-mod, $\operatorname{Tor}_1^A(M,N) = 0$

Lemma 2.1. M mod over A noetherian is flat

$\iff \operatorname{Tor}_1^A(M, A/p) = 0 \ \forall\, p \in \operatorname{Spec} A$.

Prop. 2.2. If $A' \to A$ surjective morphism of noeth. ring. with kernel J s.t. $J^2 = 0$, then M' is flat over A' iff

(1) $M = M' \otimes_{A'} A$ flat over A

(2) $M \otimes_A J \to M'$ injective

Proof. Since $J^2 = 0$, J is an $A'/J \simeq A$-mod and
$$M \otimes_A J \simeq (M' \otimes_{A'} A) \otimes_A J \simeq M' \otimes_{A'} (A \otimes A J)$$
$$\simeq M' \otimes_{A'} J.$$

Assume M' is flat over A'. (1) is base change ✓

For (2) take $0 \to J \to A' \to A \to 0$

$$\rightsquigarrow 0 \to \underset{\substack{\wr\wr \\ M \otimes_A J}}{M' \otimes_{A'} J} \to \underset{\substack{\wr\wr \\ M'}}{M' \otimes_{A'} A'} \to M' \otimes_{A'} A \to 0$$

is exact.

Suppose (1) and (2) are true. By Lemma 2.1 we can prove $\mathrm{Tor}_1^{A'}(M', A'/\mathfrak{p}')$ $\forall\, \mathfrak{p}' \in \mathrm{Spec}(A')$.
J is nilpotent $\rightsquigarrow J \subseteq \mathfrak{p}'$, take $\mathfrak{p} := \mathfrak{p}'/J \in \mathrm{Spec}\, A$, then

the following has exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] \arrow[d, equal] & \mathfrak{p}' \arrow[r] \arrow[d] & \mathfrak{p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & A'/\mathfrak{p}' \arrow[r] \arrow[d] & A/\mathfrak{p} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

then doing $\otimes M'$

$$\begin{tikzcd}
 & & \cancel{\mathrm{Tor}_1^{A'}(M', A')} \to 0 \arrow[d] & \cancel{\mathrm{Tor}_1^{A}(M, A)} \to 0 \arrow[d] & \\
 & & \mathrm{Tor}_1^{A'}(M', A'/\mathfrak{p}) \arrow[r] \arrow[d] & \cancel{\mathrm{Tor}_1^{A}(M, A/\mathfrak{p})} \to 0 \arrow[d] & \\
 & \boxed{M \otimes_A J = M' \otimes_{A'} J} \arrow[r] \arrow[d, equal, "f"] & M' \otimes_{A'} \mathfrak{p}' \arrow[r] \arrow[d, "g"] & M' \otimes_{A'} \mathfrak{p} \simeq M \otimes_A \mathfrak{p} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r, "(2)"] & M' \otimes_{A'} J \arrow[r] & M' \arrow[r] \arrow[d] & M' \otimes_{A'} A = M \arrow[r] \arrow[d] & 0 \\
 & & M' \otimes_{A'} A'/\mathfrak{p}' \arrow[r] \arrow[d] & M' \otimes_{A'} A/\mathfrak{p} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

$\simeq M \otimes_A A/\mathfrak{p}$

Using snake lemma in the rectangle

$$\overset{\to 0}{\cancel{\ker f}} \to \ker g \to \ker h \to \overset{\to 0}{\cancel{\operatorname{coker} f}} \to \operatorname{coker} g \to \operatorname{coker} h$$

$$\ker g = \operatorname{Tor}_1^{A'}(M', A'/\mathfrak{p}'), \quad \ker h = \operatorname{Tor}_1^{A}(M, A/\mathfrak{p})$$

$$\rightsquigarrow \operatorname{Tor}_1^{A'}(M', A'/\mathfrak{p}') = \operatorname{Tor}_1^{A'}(M, A/\mathfrak{p}) \overset{(1)}{=} 0$$

■

§2. Deformations over D

X will be a sch./k, $Y \subseteq X$ closed subsch.

Def. A def. of Y over D is a closed subscheme $Y' \subseteq X \times_k D$, flat over D s.t. $Y' \times_D k \simeq Y$, i.e.,

$$\begin{tikzcd} f^{-1}(\operatorname{Spec} k) \simeq Y = Y' \times_D k \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ Y' \arrow[r, "f"] & \operatorname{Spec} D \end{tikzcd}$$

(scheme-theoretic fiber over a k-point)

[There are two structural morphisms $k \hookrightarrow D$, $D \twoheadrightarrow k$, $(a + bt \mapsto a)$ (red. mod t).]

Affine case: $X = \operatorname{Spec} B$ with B a k-alg.,

$Y = \operatorname{Spec} B/I$, $X \times_k D = \operatorname{Spec} B'$ with

$$B' := B \otimes_k D \simeq B[t]/\langle t^2 \rangle.$$

We want to find $I' \subseteq B'$ s.t. B'/I' is flat over D and $(B'/I') \otimes_D k = B/I$.

Remark. Note that there is a map

$$\varphi : B' \simeq B[t]/t^2 \twoheadrightarrow B \quad \text{(reduction mod } t\text{)}$$
$$a + bt \longmapsto a$$

that induces $B'/tB' \simeq B$, and last condition is equivalent to say that image of $I' \subseteq B'$ in B is I.

By Prop. 2.2 and since B is flat over k (free because k is a field), B'/I' flat over D is equivalent to the injectivity of the map
(take $A' = D$, $A = k$, $M = B/I$, $M' = B'/I'$, $J = \langle t \rangle$)

$$B/I \simeq (B/I) \otimes_k \langle t \rangle \longrightarrow B'/I', \quad b \otimes t \mapsto tb$$

so B'/I' being flat over D is equivalent to the exactness of

$$0 \to B/I \xrightarrow{\times t} B'/I' \to B/I \to 0$$

$$(B'/I')/t(B'/I') \simeq (B'/I') \otimes_D D/\langle t \rangle \simeq (B'/I') \otimes_D k \simeq B/I$$

Take I such ideal, then there is a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\times t"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "\times t"] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/I \arrow[r, "\times t"] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Columns are exact, and exactness of bottom row implies exactness of top row (9-lemma).

Prop. 2.3. Give $I' \subseteq B'$ s.t. B'/I' flat over D and $B'/I' \otimes_D k = B/I$ is equivalent to $\varphi \in \mathrm{Hom}_B(I, B/I)$.

Proof. Note that $B' \simeq B \oplus tB$, so B' is naturally a B-mod. For such an ideal I', take $x \in I$ and a lift $x' \in I'$, and note that $x' = x + ty$ for some $y \in B$. If x', x'' are two lifts, then

$$x' - x'' = t(y - y') \in I' \rightsquigarrow y - y' \in I \text{ (flatness of } B'/I').$$

This defines a morphism $\varphi : I \to B/I$, $x \mapsto [y]$.

Now, take $\varphi \in \mathrm{Hom}_B(I, B/I)$, and

$$I' = \{x + ty \mid x \in I,\ y \in B,\ y = \varphi(x) \text{ in } B/I\}.$$

It is direct to check this is an ideal, that image in B is I and

$$0 \to I \xrightarrow{\times t} I' \to I \to 0$$

is exact. ∎

"Gluing" these arguments, we can obtain a global statement.

Thm 2.4. X sch. $/k$, $Y \subseteq X$ closed. Def. of Y/D in X. are in correspondence with $H^0(Y, \mathcal{N}_{Y/X})$.

Corollary 2.5. If $X = \mathbb{P}^n_k$, tangent space of Hilbert sch. at $[Y] \in H$ is isomorphic to $H^0(Y, \mathcal{N}_{Y/X})$.

Proof. $T_{[Y]}H \simeq \{ f : D \to H,\ f(D) = [Y] \}$

$$\simeq \{ Y' \subseteq X \times_k D \text{ closed flat s.t. } Y' \times_D k = Y \}$$

univ. prop. of H

$$= H^0(Y, \mathcal{N}_{Y/X}) \quad \blacksquare$$

§3. Deformations of line bundles and coh. sheaves / D

Def. Let $\mathcal{F} \in \underline{\mathrm{Coh}}(X)$, X sch. $/k$. A deformation of X over D is a coh. sheaf $\mathcal{F}'$ on $X' = X \times_k D$, flat over D, s.t. $\mathcal{F}' \otimes_D k \simeq \mathcal{F}$.

(Here, $\mathcal{F}' \otimes_D k$ means $\iota^* \mathcal{F}'$ where $\iota : X \hookrightarrow X'$).

We say two defs. $\mathcal{F}_1'$, $\mathcal{F}_2'$ of $\mathcal{F}$ over D are equiv. if $\mathcal{F}_1' \simeq \mathcal{F}_2'$ s.t.

$$\begin{tikzcd} \mathcal{F}_1' \otimes_D k \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{F}_2' \otimes_D k \arrow[dl, "\sim"] \\ & \mathcal{F} & \end{tikzcd}$$

commutes

Prop. (1) Isomorphism classes of $\mathcal{L}'$ on X' s.t. $\mathcal{L}' \otimes \mathcal{O}_X \simeq \mathcal{L}$ (same as $\mathcal{L}' \otimes_D k$) are given by $H^1(X, \mathcal{O}_X)$

(2) Eq. classes of def. of $\mathcal{F}/D$ is in correspondence with $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$.

Corollary. $\mathcal{E} \to X$ v.b., then defs. of $\mathcal{E}$ are given by $H^1(X, \mathcal{E}nd\,\mathcal{E})$.

Proof. $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \simeq \operatorname{Ext}^1(\mathcal{O}_X, \underbrace{\mathcal{E}^\vee \otimes \mathcal{E}}_{\simeq \mathcal{E}nd\,\mathcal{E}}) \simeq H^1(X, \mathcal{E}nd\,\mathcal{E})$ $\blacksquare$

Example. We know that every v.b on $\mathbb{P}^1$ is $\mathcal{E} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some $a_i \in \mathbb{Z}$. In part. every rank 2, deg. 0 v.b. is of the form $\mathcal{O}(a) \oplus \mathcal{O}(-a)$, $a \in \mathbb{Z}$. However, for $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ we can notice that

$$\mathcal{E}nd(\mathcal{E}) \simeq \mathcal{E} \otimes \mathcal{E}^\vee = (\mathcal{O}(1) \oplus \mathcal{O}(-1))^{\otimes 2}$$
$$\simeq \mathcal{O}(2) \oplus \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-2)$$

$$\rightsquigarrow H^1(\mathbb{P}^1, \mathcal{E}nd\,\mathcal{E}) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) = \mathbb{C},$$

so there exists a 1-dim. family of deformations of $\mathcal{E}$ over D. A non-trivial def. corresponds to a non-split extension

$$0 \to \mathcal{O}(-1) \to \mathcal{E}_t = \mathcal{O}(a) \oplus \mathcal{O}(-a) \to \mathcal{O}(1) \to 0$$

Taking coh.

$$0 \to \underbrace{H^0(\mathcal{O}(-1))}_{=0} \to H^0(\mathcal{O}(a)) \oplus H^0(\mathcal{O}(-a)) \to \underbrace{H^0(\mathcal{O}(1))}_{=\mathbb{C}^2} \to \underbrace{H^1(\mathcal{O}(-1))}_{=0}$$

$$\to H^1(\mathcal{O}(a)) \oplus H^1(\mathcal{O}(-a)) \to \underbrace{H^1(\mathcal{O}(1))}_{=0} \to 0$$

$$\rightsquigarrow \quad h^0(\mathcal{O}(a)) + h^0(\mathcal{O}(-a)) = 2$$

$$h^1(\mathcal{O}(a)) = h^1(\mathcal{O}(-a)) = 0 \rightsquigarrow a \in \{0,1\}$$

$$\rightsquigarrow \mathcal{E}_t \simeq \mathcal{O} \oplus \mathcal{O}.$$

Section 3. T^i functors

Construction. Let A, B rings with $A \to B$ morphism, M a B-mod. We will define some functors $T^i(B/A, M)$ for $i = 0, 1, 2$.

Take $R = A[x]$ for a set of vars. $x = \{x_i\}$, then there is an exact sequence

$$0 \to I \longrightarrow R \longrightarrow B \longrightarrow 0$$

for some ideal I. Choose also F a free R-mod with $j: F \twoheadrightarrow I$, $Q := \ker j$, then

$$0 \to Q \to F \xrightarrow{j} I \longrightarrow 0$$

Now, define

$$F_0 = \left\{ \begin{array}{l} \text{submod. of } F \text{ generated by} \\ j(a)b - j(b)a \text{ for } a, b \in F \end{array} \right\}$$ "Koszul relations"

By def. $j(F_0) = 0$, i.e., $F_0 \subseteq Q$. We will define a complex of B-modules

$$L_2 \xrightarrow{d_2} L_1 \longrightarrow L_0$$

with this information.

Take $L_2 = Q/F_0$ and note L_2 is a B-mod.

Proof: Let $x \in I$, $a \in Q$, then $x = j(x')$, $x' \in F$ and $xa = j(x')a \equiv x' \underbrace{j(a)}_{=0} \mod F_0$. $\blacksquare$

Now, define $L_1 = F \otimes_R B = F \otimes_R R/I = F/IF$ and $d_2 : L_2 \to L_1$ induced by

$$Q \hookrightarrow F \twoheadrightarrow IF$$

Finally, define

$$L_0 = \Omega_{R/A} \otimes_R B,$$

where $\Omega_{R/A}$ is the mod. of rel. diff., defined as

$$\Omega_{R/A} = \langle dr \mid r \in R \rangle_{\text{free } A\text{-mod}} \Big/ \left\{ \begin{array}{l} d(r+r') = dr + dr' \\ d(rr') = r dr' + r' dr \\ d(a) = 0, \ a \in A \end{array} \right\}$$

with a univ. morphism

$$\begin{aligned} d : R &\to \Omega_{R/A} \\ r &\mapsto dr \end{aligned}$$

Note that there is an induced map $\varphi : L_1 := F/IF \to I/I^2$ and an exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{R/A} \otimes_R B \longmapsto \Omega_{B/A} \to 0$$
$$r \longmapsto dr \otimes 1$$

so, we define $d_1 := \delta \circ \varphi : L_1 \to L_0$.

Remark. $d_1 d_2 (q + F_0) = d_1 (q + IF) = 0$
$j(q) = 0$

Thus, $L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$ is a complex of B-mods.

Remark. $L_1 = F \otimes_R B$ free B-mod.
$L_0 = \Omega_{R/A} \otimes B$ free B-mod (since $R = A[x]$).

Def. For $M \in B$-Mod, we define

$$T^i(B/A, M) = h^i(\mathrm{Hom}_B(L_\bullet, M))$$

Explicitely, $L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$ induces

$$\mathrm{Hom}_B(L_2, M) \xleftarrow{d_2^*} \mathrm{Hom}_B(L_1, M) \xleftarrow{d_1^*} \mathrm{Hom}_B(L_0, M)$$
$$f d_1 \longleftarrow f$$
$$g d_2 \longleftarrow g$$

Then,
$$T^0(B/A, M) = \ker d_1^*$$
$$T^1(B/A, M) = \ker d_2^* / \mathrm{Im}\, d_1^*$$
$$T^2(B/A, M) = \mathrm{Hom}_B(L_2, M) / \mathrm{Im}\, d_2^*$$

Claim. The modules T^i are independents of the choice of F and R.

Properties derived from the construction.

(1) For $A \to B$, $M \in B\text{-}\underline{Mod}$,

$$L_1 = F/IF \twoheadrightarrow I/I^2 \,, \quad \underbrace{L_1 \xrightarrow{d_1} L_0 \to \Omega_{B/A} \to 0}_{\text{exact}}$$

Taking $\mathrm{Hom}_B(-, M)$ we have

$$\mathrm{Hom}_B(L_1, M) \xleftarrow{d_1^*} \mathrm{Hom}_B(L_0, M) \leftarrow \underset{\mathrm{Der}_A(B, M)}{\underset{\wr\wr}{\mathrm{Hom}_B(\Omega_{B/A}, M)}} \leftarrow 0$$

$$\Rightarrow T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M)$$

In particular, $T^0(B/A, B) = \mathrm{Hom}_B(\Omega_{B/A}, B) =: T_{B/A}$

"the tangent mod. of B over A"

(2) If $B = A[x]$, take $R = B$, $I = 0$, $F = 0$,

so $L_2 = L_1 = 0 \rightsquigarrow T^i = 0$ for $i = 1, 2$, $\forall M$.

(3) Suppose $f : A \to B$ is surjective, $I = \ker f$. Then,

$T^0(B/A, M) = 0$ and $T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M)$ $\forall M$.

Indeed, take $R = A \Rightarrow L_0 = \Omega_{A/A} \otimes B = 0 \Rightarrow T_0 = 0$.

*) $0 \to Q \to F \to I \to 0$ gives

$$Q \otimes_A B \to F \otimes_A B \to \overset{\overset{I/I^2}{\shortparallel}}{I \otimes_A B} \to 0$$

*) $Q \twoheadrightarrow Q/F_0 \Rightarrow Q \otimes_A B \twoheadrightarrow (Q/F_0) \otimes_A B \simeq Q/F_0$

Then,

$$\begin{tikzcd} Q \otimes_A B \arrow[r] \arrow[d] & L_1 \arrow[r] & I/I^2 \arrow[r] & 0 \\ L_2 = Q/F_0 \arrow[ur] & & & \end{tikzcd}$$

gives us

$$\mathrm{Hom}_B(L_2, M) \xleftarrow{d_2^*} \mathrm{Hom}_B(L_1, M) \xleftarrow{d_1^*} \mathrm{Hom}_B(I/I^2, M) \leftarrow 0$$

$$\rightsquigarrow \quad T^1(B/A, M) = \ker d_2^* / \underbrace{\mathrm{Im}\, d_1^*}_{= 0} = \mathrm{Hom}_B(I/I^2, M)$$

(4) By construction, $T^1(B/A, -) : B\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is a covariant additive functor, and if $0 \to M' \to M \to M'' \to 0$ is exact, then there is a l.e.s

$$\begin{aligned} 0 &\to T^0(B/A, M') \to T^0(B/A, M) \to T^0(B/A, M'') \\ &\to T^1(B/A, M') \to T^1(B/A, M) \to T^1(B/A, M'') \\ &\to T^2(B/A, M') \to T^2(B/A, M) \to T^2(B/A, M''). \end{aligned}$$

Thm. If $A \to B \to C$ are rings, $M \in C\text{-}\underline{\mathrm{mod}}$, then there is an exact sequence

$$\begin{aligned} 0 &\to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M) \\ &\to T^1(C/B, M) \to T^1(C/A, M) \to T^1(B/A, M) \\ &\to T^2(C/B, M) \to T^2(C/A, M) \to T^2(B/A, M). \end{aligned}$$

Corollary. Suppose $A = k[x_1, \dots, x_n]$, $B = A/I$. Then, $\forall M$, the sequence

$$0 \to T^0(B/k, M) \to \mathrm{Hom}(\Omega_{A/k}, M) \to \mathrm{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0$$

is exact and $T^2(B/A, M) \simeq T^2(B/k, M)$.

Proof. $k \to A \to B$ induces the exact sequence

$$
\begin{aligned}
0 \to \cancel{T^0(B/A, M)}^{(3)} \to T^0(B/k, M) \to\ & T^0(A/k, M) \overset{(1)}{\cong} \mathrm{Hom}(\Omega_{A/k}, M) \\
\to T^1(B/A, M) \to T^1(B/k, M) \to\ & \cancel{T^1(A/k, M)}^{(2)} \\
\overset{(3)}{\cong} \mathrm{Hom}(I/I^2, M) \qquad & \\
\to T^2(B/A, M) \to T^2(B/k, M) \to\ & \cancel{T^2(A/k, M)}^{(2)}
\end{aligned}
$$

$\blacksquare$