Talk 4.2

§1. Artin rings

Now, we will consider "higher-order" defs., which are defs. over arbitrary rings.

Def. A small extension is an exact sequence

$$0 \to J \to C' \to C \to 0$$

where C, C' are local artin rings, $C/\mathfrak{m}_C = k$ and $\mathfrak{m}_{C'} J = 0$.

In part., J is k-vector space, and since $J^2 = 0$, J is a C-module.

We say this extension is split if it is isomorphic to $J \oplus C$, i.e., there is a diagram

$$\begin{tikzcd} & & J \oplus C \arrow[dr] & & \\ 0 \arrow[r] & J \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & & C' \arrow[uu, "s"] \arrow[ur] & & \end{tikzcd}$$

Example.

$$0 \to \langle t^n \rangle \to k[t]/\langle t^{n+1} \rangle \to k[t]/\langle t^n \rangle \to 0$$

Notice that for $n = 1$, the sequence splits:

$$0 \to \langle t \rangle \to k[t]/\langle t^2 \rangle \to k \to 0$$

Theorem. Every local Artin k-alg can be obtained by small extensions starting from k. In particular, every surjection $C' \to C$ is a composition of extensions.

This theorem gives the following idea: if we have a structure S (scheme, sheaf, etc) over C, for $C' \to C$, we try to classify defs. $/C'$ s.t. restricts to the original one.

§2. Deformations

Recall. For X_0 sch. $/k$, a def. over an Artin ring C is a scheme X flat over C s.t. $X_0 \simeq X \times_C k$.

We will try to understand deformations over the ring $A_n = k[t]/\langle t^{n+1} \rangle$ lifting from $A_1 = D$.

Situation. Take a small extension

$$0 \to J \to C' \to C \to 0$$

Consider X a def. of X_0, $Y_0 \subseteq X_0$ a closed subscheme and $Y \subseteq X$ a closed subscheme s.t. $Y \times_C k \simeq Y_0$. Now, we consider X' a def. of X over C', i.e., X' flat over C' s.t. $X' \times_{C'} C \simeq X$.

We aim to classify $Y' \subseteq X'$ closed subsch. flat over C' s.t. $Y \simeq Y' \times_{C'} C$.

The situation is like the following diagram

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[d, hook] & Y \arrow[r, hook] \arrow[d, hook] & Y' \arrow[d, hook] & \text{¿?} \arrow[l] \\ X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ k \arrow[r] & C \arrow[r] & C' \end{tikzcd}$$

Def. Let G be a group acting on a set S. We say S is a G-torsor if $\exists\ s_0 \in S$ that $G \to S$, $g \mapsto g \cdot s_0$ is a bijection. S is a pseudotorsor if we allow $S = \phi$.

Thm. In the previous situation:

(a) Extensions of Y over C' in X' is a pseudotorsor under the action of $H^0(Y_0, N_{Y_0/X_0} \otimes J)$.

(b) If there is a local extension over C', there is an obstruction $\alpha \in H^1(Y_0, N_{Y_0/X_0} \otimes J)$ that vanishes iff there is a global deformation Y'.

Sketch of the proof.

(a) Affine case: $X = \operatorname{Spec} A$, $X' = \operatorname{Spec} A'$,

$Y = \operatorname{Spec} B = \operatorname{Spec} A/I$.

Situation

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \otimes_C I \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \otimes_C A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \otimes_C B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We want to find B' and I' !!

(*) Exactness of bottom and middle rows are equivalent to flatness of B', A' over C'.

(**) Left column is exact because of the flatness of B over C.

Idea: to give I', I'' fitting in the diagram is the same as giving $\varphi \in \mathrm{Hom}_A(I, J \otimes_C B)$, then the association of $(I', \varphi) \mapsto I''$ defines a group action of $\mathrm{Hom}_A(I, J \otimes_C B)$ on the set of ideals I'.

Notice that

$$\mathrm{Hom}_A(I, J\otimes_C B) \simeq \mathrm{Hom}_A(I, J\otimes_k B_0)$$

$$J\otimes_C B \simeq (J\otimes_k k)\otimes_C B \simeq J\otimes_k (k\otimes_C B) \simeq J\otimes_k B_0$$

$$Y_0 = \mathrm{Spec}\, A_0/I_0$$

$$A_0 \to A$$

Local to global: $\mathrm{Hom}_{A_0}(I_0, J\otimes_k B_0) \rightsquigarrow H^0(Y_0, N_{Y_0/X_0}\otimes J)$

(b) Suppose $\exists$ (U_i) an open affine covering of X s.t. there exists Y_i' def. of $Y\cap U_i$ in $U_i' := U_i \times_C C' \subseteq X'$. On $U_{ij} = U_i \cap U_j$ we have two extensions $Y_i' \cap U_{ij}$ and $Y_j' \cap U_{ij}$. By (a) this defines $\alpha_{ij} \in H^0(U_{ij}, N_{Y_0/X_0}\otimes J)$, and over U_{ijk} there are three defs. Y_i', Y_j', Y_k' defining $\alpha_{ij}, \alpha_{jk}, \alpha_{ik} \in H^0(U_{ij}, N_{Y_0/X_0}\otimes J)$ that satisfies $\alpha_{ik} = \alpha_{ij} + \alpha_{jk}$. Moreover, if Y_i'' is another set of defs., Y_i', Y_i'' define $\beta_i \in H^0(U_i, N_{Y_0/X_0}\otimes J)$ and notice that $\alpha_{ij}'' = \alpha_{ij} + \beta_j - \beta_i$ $\rightsquigarrow$ this defines $\alpha \in H^1(Y_0, N_0 \otimes J)$ depending just on Y.

Then we can see $\alpha = 0$ iff $\exists\, Y'$ global def.
If Y' exists, take $Y_i' = Y' \cap U_i'$, so $\alpha_{ij} = 0$
$\rightsquigarrow \alpha = 0$. If $\alpha = 0$ in $H^1(Y_0, \mathcal{N}_{Y_0/X_0} \otimes J)$,
then $\alpha_{ij} = \beta_i - \beta_j$ ~~for~~ for some $\beta_i \in H^1(U_i, \mathcal{N}_{Y_0/X_0} \otimes J)$.
Take Y_i'' def. corresponding to β_i on U_i (i.e., modify Y_i' to Y_i''), then, notice that in U_{ij} differences are

$$\alpha_{ij}'' = \alpha_{ij} + \beta_j - \beta_i = 0 \qquad \blacksquare$$

Example. Let X_0 be a smooth proj. surface and Y_0 a curve s.t. $Y_0 \simeq \mathbb{P}^1$ and $Y_0^2 = -1$. Then we know $\mathcal{N}_{Y_0/X_0} \simeq \mathcal{O}(-1)$

$$\rightarrow H^0(Y_0, \mathcal{N}_{Y_0/X_0}) = H^1(Y_0, \mathcal{N}_{Y_0/X_0}) = 0$$

and there no obstruction to lift deformations.
(Idea: we can blow-down the curve and move the base point.)

Corollary. Let Y be closed subsch. of $X = \mathbb{P}^n_k$ s.t. $H^1(Y, \mathcal{N}_{Y/X}) = 0$. Then, Hilbert scheme is non-singular at $[Y]$.