Calculating the homology of D_3

Facts: ① Given a subgroup $H \leq G$ we can have models for the classifying spaces s.t. $BH \to BG$ is a $[G:H]$-fold cover, provided that H is finite index

② Given a covering $\pi : \tilde{X} \to X$ which is the quotient of a $G \curvearrowright \tilde{X}$ for $|G| = n$, if k is a field s.t. $\mathrm{char}(k) \nmid n$, then $\pi^* : H^n(X;k) \to H^n(\tilde{X};k)$ is an inclusion with image $H^n(\tilde{X};k)^G$.

(Prop. 3G. 1. Hatcher)

③ In the setting of Fact ②, there is a $\tau : \tilde{H}_*(X;\mathbb{Z}) \to \tilde{H}_*(\tilde{X};\mathbb{Z})$ s.t. $\tilde{H}_*(X;\mathbb{Z}) \xrightarrow{\tau} \tilde{H}_*(\tilde{X};\mathbb{Z}) \xrightarrow{\pi_*} \tilde{H}_*(X;\mathbb{Z})$ is the $\cdot n$ map.

④ $H^*(\mathbb{Z}/3;\mathbb{F}_3) \cong \mathbb{F}_3[\sigma,\beta]/(\sigma^2)$ $\quad |\sigma| = 1,\ |\beta| = 2$

(Ex. 6.7.5. Weibel)

$D_3 = \mathbb{Z}/3 \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2 \curvearrowright \mathbb{Z}/3$ via $\cdot(-1)$.

$H^*(D_3;\mathbb{F}_2) \cong H^*(\mathbb{Z}/2;\mathbb{F}_2)^{\mathbb{Z}/3}$ but here the only action possible is the trivial action:

$$H^*(\mathbb{Z}/2;\mathbb{F}_2) = H_*(\mathbb{Z}/2;\mathbb{F}_2) = H_*(\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\cdot 2} \mathbb{Z}/2 \xleftarrow{0} \ldots)$$
$$= \mathbb{Z}/2 \quad \forall * \geq 0$$

$\mathbb{Z}/3$ can only act trivially on $\mathbb{Z}/2$

$H^*(D_3;\mathbb{F}_3) \cong H^*(\mathbb{Z}/3;\mathbb{F}_3)^{\mathbb{Z}/2} \cong \mathbb{F}_3[\beta\sigma, \beta^2]$ $\quad |\beta\sigma| = 3,\ |\beta^2| = 4$

Fact (Weibel Eq. 6.7.10): $\mathbb{Z}/2 \curvearrowright \mathbb{F}_3[\sigma,\beta]/(\sigma^2)$ acts via $\sigma \mapsto -\sigma$ &

$\beta \mapsto -\beta$

Consider the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/6 \to \mathbb{Z}/3 \to 0$
this induces a LES

$$\dots \to H_i(BD_3;\mathbb{Z}/2) \to H_i(BD_3;\mathbb{Z}/6) \to H_i(BD_3;\mathbb{Z}/3) \to$$
$$\to H_{i-1}(BD_3;\mathbb{Z}/2) \to H_{i-1}(BD_3;\mathbb{Z}/6) \to H_{i-1}(BD_3;\mathbb{Z}/3) \to \dots$$

from $H^*(X;\mathbb{F}) \cong H_*(X;\mathbb{F})$ $\forall X \in \mathrm{Top}$, $\mathbb{F}$ field via UCT
we have

$$H_i(BD_3;\mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \forall i$$

$$H_i(BD_3;\mathbb{Z}/3) \cong \begin{cases} \mathbb{Z}/3 & \text{if } i \equiv 0 \text{ or } 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

thus this LES is of the form: say $i \equiv 0 \bmod 4$

$$\dots \to \mathbb{Z}/2 \to H_{i+3}(D_3;\mathbb{Z}/6) \to \mathbb{Z}/3 \to$$
$$\to \mathbb{Z}/2 \to H_{i+2}(D_3;\mathbb{Z}/6) \to 0 \to$$
$$\to \mathbb{Z}/2 \to H_{i+1}(D_3;\mathbb{Z}/6) \to 0 \to$$
$$\to \mathbb{Z}/2 \to H_i(D_3;\mathbb{Z}/6) \to \mathbb{Z}/3 \to \dots$$

since the only map $\mathbb{Z}/3 \to \mathbb{Z}/2$ is trivial and
$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/3) \cong \mathbb{Z}/\gcd(2,3) \cong 0$ we must have

$$H_i(D_3;\mathbb{Z}/6) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i \equiv 1 \text{ or } 2 \bmod 4 \\ \mathbb{Z}/6 & \text{if } i \equiv 0 \text{ or } 3 \bmod 4 \end{cases}$$

Now since

$$\begin{tikzcd} \tilde{H}^i(BD_3;\mathbb{Z}) \arrow[r, "\iota"] \arrow[rr, bend right=40, "\cdot 6"'] & \tilde{H}^i(B1;\mathbb{Z}) \arrow[r, "\pi^*"] \arrow[d, equal] & \tilde{H}^i(BD_3;\mathbb{Z}) \\ & 0 & \end{tikzcd}$$

the $\cdot 6$ map annihilates $\tilde{H}^i(BD_3;\mathbb{Z})$. Applying the universal
coefficient theorem we obtain:

$$0 \to \underbrace{H_i(BP_3;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/6}_{\substack{\cong \\ \operatorname{coker}(H_i(BD_3;\mathbb{Z}) \xrightarrow{\cdot 6} H_i(BD_3;\mathbb{Z}))}} \to H_i(BD_3;\mathbb{Z}/6) \to \underbrace{\operatorname{Tor}^1_{\mathbb{Z}}(H_{i-1}(BD_3;\mathbb{Z});\mathbb{Z}/6)}_{\substack{\cong \\ \ker(H_{i-1}(BD_3;\mathbb{Z}) \xrightarrow{\cdot 6} H_{i-1}(BD_3;\mathbb{Z}))}} \to 0$$

Clearly $H_0(BD_3;\mathbb{Z}) \cong \mathbb{Z}$

$$0 \to \underbrace{H_1(BP_3;\mathbb{Z})}_{\mathbb{Z}/2} \xrightarrow{\quad\Leftarrow\quad} \underbrace{H_1(BD_3;\mathbb{Z}/6)}_{\mathbb{Z}/2} \to \underbrace{\operatorname{Tor}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/6)}_{0} \to 0$$

$$0 \to \underbrace{H_2(BD_3;\mathbb{Z})}_{0} \xrightarrow{\quad\Leftarrow\quad} \underbrace{H_2(BD_3;\mathbb{Z}/6)}_{\mathbb{Z}/2} \to \underbrace{\operatorname{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/6)}_{\mathbb{Z}/2} \to 0$$

$$0 \to \underbrace{H_3(BD_3;\mathbb{Z})}_{\mathbb{Z}/6} \xrightarrow{\quad\Leftarrow\quad} \underbrace{H_3(BP_3;\mathbb{Z}/6)}_{\mathbb{Z}/6} \to \underbrace{\operatorname{Tor}^1_{\mathbb{Z}}(0,\mathbb{Z}/6)}_{0} \to 0$$

$$0 \to \underbrace{H_4(BD_3;\mathbb{Z})}_{0} \xrightarrow{\quad\Leftarrow\quad} \underbrace{H_4(BD_3;\mathbb{Z}/6)}_{\mathbb{Z}/6} \to \underbrace{\operatorname{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/6,\mathbb{Z}/6)}_{\mathbb{Z}/6} \to 0$$

$$0 \to \underbrace{H_5(BP_3;\mathbb{Z})}_{\mathbb{Z}/2} \xrightarrow{\quad\Leftarrow\quad} \underbrace{H_5(BP_3;\mathbb{Z}/6)}_{\mathbb{Z}/2} \to \underbrace{\operatorname{Tor}^1_{\mathbb{Z}}(0,\mathbb{Z}/6)}_{0} \to 0$$

induction now yields:

$$H_i(D_3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i=0 \\ \mathbb{Z}/2 & \text{if } i \equiv 1 \bmod 4 \\ \mathbb{Z}/6 & \text{if } i \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

If one is willing to learn about twisted coefficients this follows from s.seq. see Weibel Eg. 6.8.5.