Why study thermal CFTs?

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1 What is a thermal CFT?

- An alternative way to formulate quantum mechanics is to use density matrices, e.g. $\hat{\rho} = |\Psi\rangle\langle\Psi|$, instead of Hilbert space vectors $|\Psi\rangle$. Then an expectation value is defined as

$$\langle\mathcal{O}\rangle = \langle\Psi|\hat{\mathcal{O}}|\Psi\rangle = \sum_i \langle\psi_i|\Big(|\Psi\rangle\langle\Psi|\Big)\hat{\mathcal{O}}|\psi_i\rangle = \sum_i \langle\psi_i|\hat{\rho}\hat{\mathcal{O}}|\psi_i\rangle \equiv \mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}}) \tag{1}$$

- This can also accommodate cases where you don't know the exact Hilbert space state of the system: take linear combinations of density matrices. Then the expectation value is as if you're in any particular state with some probability:

$$\langle\mathcal{O}\rangle = \mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}}) = \sum_i \mathrm{Tr}(c_i\hat{\rho}_i\hat{\mathcal{O}}) = \sum_i c_i\langle\Psi_i|\hat{\mathcal{O}}|\Psi_i\rangle \tag{2}$$

- This is ideal for a system at a given temperature, since then the microstate is unknown (too many degrees of freedom, too many fluctuations). Instead, we average over all possible energy states with a Boltzman factor as weight. For a quantum system at temperature $T = \beta^{-1}$, this means that

$$\hat{\rho} = e^{-\beta\hat{H}} \tag{3}$$

- Then, because always $\hat{\mathcal{O}}(t) = e^{i\hat{H}t}\hat{\mathcal{O}}(0)e^{-i\hat{H}t}$, we get that (with $\Delta t = t_2 - t_1$):

$$\begin{aligned}
\langle\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\rangle_\beta \equiv&\ \mathrm{Tr}\left(e^{-\beta\hat{H}}e^{i\hat{H}t_1}\hat{\mathcal{O}}_1e^{-i\hat{H}t_1}e^{i\hat{H}t_2}\hat{\mathcal{O}}_2e^{-i\hat{H}t_2}\right)\\
=&\ \mathrm{Tr}\left(\hat{\mathcal{O}}_1e^{i\hat{H}\Delta t}\hat{\mathcal{O}}_2e^{-i\hat{H}(\Delta t - i\beta)}\right)\\
=&\ \mathrm{Tr}\left(\hat{\mathcal{O}}_2e^{-i\hat{H}(\Delta t - i\beta)}\hat{\mathcal{O}}_1e^{i\hat{H}(\Delta t - i\beta)}e^{-\beta\hat{H}}\right)\\
=&\ \langle\mathcal{O}_2(t_2 - i\beta)\mathcal{O}_1(t_1)\rangle_\beta = \langle\mathcal{O}_2(t_2)\mathcal{O}_1(t_1 + i\beta)\rangle_\beta
\end{aligned} \tag{4}$$

- In other words, if we Wick rotate to $\tau = it$ and assume that $\tau_1 > \tau_2$ to ensure convergence, the Euclidean two-point function $g(\tau_1, \tau_2) = g(\tau)$ with $\tau = \tau_1 - \tau_2$ should obey

$$g(\tau) = g(\tau + \beta) \tag{5}$$

- This is known as the *Kubo-Martin-Schwinger* (KMS) condition. Axiomatically, it defines what we mean by a thermal state; a thermal CFT is simply a CFT in a thermal state.

- In the Euclidean case, you can interpret the KMS condition as a compactification in one direction: $\mathbb{R}^d$ becomes $S^1 \times \mathbb{R}^{d-1}$.

- For CFTs, we are used to working in Euclidean signature; additionally, we often compactify the spatial directions as well (to combat IR divergences). Thus, thermal CFTs are equivalent to CFTs on a compact manifold $S^1 \times S^{d-1}$. The radius of S^{d-1} is often fixed.

2 Why study thermal CFTs? Holography

- A theoretically very important reason to study thermal CFTs is because some of them are holographic, in which case they correspond to black holes in AdS_{d+1}.
- How do we see this? First, intuitively: a thermal state has many particles (to do proper statistics), so a thermal CFT corresponds to many particles in the AdS_{d+1} bulk. Because of the potential, they gather around the origin and collapse into a black hole.
- Secondly, we argue more rigorously the other way around. Start with AdS_{d+1}-Schwarzschild spacetime, and ask what it is dual to.
- The (Euclidean) metric is

$$\mathrm{d}s^2 = V\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{V} + r^2\mathrm{d}\Omega_{d-1} \tag{6}$$

  with the following blackening function:

$$V = 1 + \frac{r^2}{L^2} - \frac{\omega_d M}{r^{d-2}} \tag{7}$$

  where ω_d is a dimensionful constant that depends on d (through e.g. Newton's constant).
- A horizon occurs when $V = 0$; the event horizon is at r_+, the largest r for which this happens, and forms the edge of spacetime in Euclidean signature (afterwards, the metric signature is indeterminate).
- If $\delta r = r - r_+ > 0$ is small (i.e. $\delta r \ll r_+$), then the metric becomes

$$\begin{aligned}\mathrm{d}s^2 &= V'(r_+)\delta r\mathrm{d}\tau^2 + \frac{\mathrm{d}\delta r^2}{V'(r_+)\delta r} + (\delta r + r_+)^2\mathrm{d}\Omega_{d-1} \\ &= \frac{[V'(r_+)]^2}{4}x^2\mathrm{d}\tau^2 + \mathrm{d}x^2 + \left(\frac{1}{4}V'(r_+)x^2 + r_+\right)^2\mathrm{d}\Omega_{d-1}\end{aligned} \tag{8}$$

  with $x = 2\sqrt{\delta r/V'(r_+)}$.
- The τ, x-plane should become Euclidean space for such small distances (since space is locally flat); this is only possible if $\theta \equiv \frac{1}{2}V'(r_+)\tau \in [0, 2\pi)$. Thus, the periodicity of τ has to be $\beta = \frac{4\pi}{V'(r_+)}$.
- The Euclidean metric at any fixed r (including $r \to \infty$) is therefore precisely $S^1 \times S^{d-1}$. At the boundary, we must have a thermal CFT! Its temperature is the radius of the S^1, which in this case is

$$\beta = \frac{4\pi}{V'(r_+)} = \frac{4\pi r_+ L^2}{r_+^2 d + (d-2)L^2} \leq \frac{2\pi L}{d-1} \tag{9}$$

  This immediately tells us that β needs to be low enough (the maximum β occurs at $r_+ = L$), i.e. the temperature needs to be sufficiently high, for the AdS_{d+1}-Schwarzschild spacetime to make sense. Otherwise, a thermal CFT is just dual to regular old AdS_{d+1}, with some hot gas inside.

3 Why study thermal CFTs? Statistical physics

- An important property of statistical physics systems is universality: near a critical point, the correlation length diverges (by definition) and hence the system is described by a CFT. Which CFT that is, depends solely on the symmetry group because it determines where the RG flow will be directed to.
- Consider for example the Ising model, in d dimensions, with Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_j \sigma_j \tag{10}$$

- For $h = 0$, you see that it has a $\mathbb{Z}_2$ symmetry. In general, we could assign a vector to each lattice site and enhance the $\mathbb{Z}_2$ symmetry to an $O(N)$ symmetry.
- This can be studied using the $O(N)$ field theory, defined with

$$S = -\int \mathrm{d}^d x \left(\frac{1}{2} (\partial \phi_i)^2 + \frac{\lambda}{4!} (\phi_i \phi_i)^2 \right) \tag{11}$$

  where a sum over the index i is implicit.
- The four-point interaction has $[\lambda] = 4 - d$, and therefore it is relevant for $d < 4$; in these cases you get an IR fixed point, and the model is in the same universality class as the ($h = 0$) Ising model.
- Close to $d = 4$ (i.e. $d = 4 - \epsilon$) you can study this fixed point perturbatively in ϵ; far away ($d = 2$ or $d = 3$) you need something else.
- A good starting point is to do a Hubbard-Stratonovich transformation, wherein you replace the action by

$$S = -\int \mathrm{d}^d x \left(\frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} \sigma \phi_i^2 - \frac{3! \sigma^2}{4\lambda} \right) \tag{12}$$

- These two actions are equivalent because you can compute the σ integral in the partition function:

$$\begin{aligned} \int \mathcal{D}[\sigma] \, e^{iS} &= e^{-\frac{i}{2} \int \mathrm{d}^d x \, (\partial \phi_i)^2} \int \mathcal{D}[\sigma] \, e^{-\int \mathrm{d}^d x \left(\frac{3!(\epsilon - i)}{4\lambda} \sigma^2 + \frac{i}{2} \phi_i^2 \sigma \right)} \\ &= e^{-\frac{i}{2} \int \mathrm{d}^d x \, (\partial \phi_i)^2} C e^{-\frac{1}{2} \frac{\phi_i^2}{2} \frac{2\lambda}{3!(\epsilon - i)} \frac{\phi_i^2}{2}} \\ &= C e^{-i \int \mathrm{d}^d x \left(\frac{1}{2} (\partial \phi_i)^2 + \frac{\lambda}{4!} (\phi_i \phi_i)^2 \right)} \end{aligned} \tag{13}$$

- Observe that the quadratic σ term has coupling $[\lambda^{-1}] = d - 4 < 0$, so it is IR-irrelevant; the fixed point is located at $\lambda \to \infty$.
- An important effect at non-zero temperatures is that ϕ_i acquires a mass. You can see this from the partition function, where you can now integrate out the ϕ_i fields. For $\lambda \to \infty$, the result is

$$Z = \int \mathcal{D}[\sigma] e^{-\frac{N}{2} \mathrm{Tr} \ln(-\partial_\mu \partial^\mu + \sigma)} \tag{14}$$

- This can be computed using a saddle-point approximation if $N \to \infty$. For a thermal state in $d = 3$, the saddle-point is located where

$$\frac{\partial}{\partial \sigma} \operatorname{Tr} \ln(-\partial_\mu \partial^\mu + \sigma) = \sum_{n \in \mathbb{Z}} \int \frac{\mathrm{d}^2 \mathbf{p}}{(2\pi)^2} \frac{1}{\omega_n^2 + \mathbf{p}^2 + \sigma} = 0 \tag{15}$$

where $\omega_n = 2\pi n/\beta$ are the eigenvalues of the Laplace operator on a circle (the compactified time dimension; for now we take the spatial dimensions to be non-compact).

- You can then do the sum, regulate the integral (e.g. by introducing a cutoff) and renormalise it; the result is that at the saddle-point, the expectation value of σ itself is

$$\langle \sigma \rangle_\beta = \frac{4}{\beta} \ln^2 \left(\frac{1 + \sqrt{5}}{2} \right) \equiv m_{\text{th}}^2 \tag{16}$$

- The interpretation as a mass follows from the two-point function:

$$\langle \phi_a \phi_b \rangle_\beta (\omega_n, \mathbf{k}) = - \frac{i \delta_{ab}}{-\omega_n^2 + \mathbf{k}^2 + \langle \sigma \rangle_\beta} \tag{17}$$

4 Further reading

- Emergent spacetime and holographic CFTs;
- Anti de Sitter space and holography (especially section 3);
- Introduction to Gauge/Gravity duality;
- Lectures on AdS/CFT from the Bottom Up;
- The Conformal Bootstrap at Finite Temperature (especially section 5.1);
- and Thermal effects in conformal field theories.