Example Questions.

III.1.1.

Example 1 from this section deals with the cyclic group.

Let $G$ be cyclic of finite order $n$. Recall this resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. $\left(\mathbb{Z}G \cong \frac{\mathbb{Z}[t]}{t^k - 1}\right)$.

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \mathbb{Z} \to 0$$

where $N = 1 + t + t^2 + \cdots + t^{n-1} = \sum_{g \in G} g$.

Let $M \in {}_{\mathbb{Z}G}\mathrm{Mod}$. Then $H_*(G,M)$ is the homology of

$$\cdots \xrightarrow{N} M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M$$

because $H_*(G,M) := H(F \otimes_G M)$ and $\mathbb{Z}G \otimes_G M \cong M$

$(g \otimes m \longmapsto 1 \otimes gm)$

And $H^*(G,M)$ is the cohomology of

$$M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M \xrightarrow{N} \cdots$$

because $H^*(G,M) := H^*(\mathrm{Hom}_{\mathbb{Z}G}(F, M))$ and $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \cong M$

$(g \mapsto m) \longmapsto g^{-1}m$

$u(g) = m \Rightarrow g \cdot u(1) = m$

We have that
$$Ng = (1 + g + \cdots + g^{n-1})g = N.$$
$$gN = g(1 + g + \cdots + g^{n-1}) = N$$

So $Ngm = Nm$ & $gNm = Nm$. So $\bar{N} : M_G \to M^G$ is well defined. (called the norm map). Note, we didn't need $G$ to be cyclic group or abelian for this to work. $N = \sum_{g \in G} g$ always has this property.

1a) Show $\ker \bar{N}$ and $\operatorname{coker} \bar{N}$ are annihilated by $|G|$.

We have $Nm = |G|m$ for $m \in M_G$
also $Nm = |G|m$ for $m \in M^G$.

let $m \in \ker(\bar{N})$, then $|G|m = Nm = 0$
let $m \in \operatorname{coker}(\bar{N})$, then $|G|m = Nm = 0 \in M^G / NM_G = \operatorname{coker}(\bar{N})$.

b) Suppose $M$ is a module of the form $M = \mathbb{Z}G \otimes A$
where $A$ is an abelian group and $g \cdot (r \otimes a) = gr \otimes a$.
($M$ an induced module).

Show $\bar{N} : M_G \to M^G$ is an iso.

We have $M_G = \mathbb{Z} \otimes A$ and $M^G = (\mathbb{Z}N) \otimes A$.

(Second point because assume $r \in \mathbb{Z}G^G$ $r = \sum_h a_h h$
$gr = \sum_h a_h gh = \sum_h a_{g^{-1}h} h \Rightarrow a_h = a_{g^{-1}h} \ \forall g \Rightarrow a_h = \text{const.}$).

Now $\bar{N} : M_G \to M^G$ is clearly a bijection. $z \otimes a \mapsto zN \otimes a$.
It is an iso by ring axioms.

c) Show $\bar{N}$ is an iso if $M$ is a projective $\mathbb{Z}G$-module.
(I needed help from soln's for this).

Step 1) Show $F \cong \bigoplus_i \mathbb{Z}G$ free module satisfies
$F_G \xrightarrow[\sim]{N} F^G$.
we have $\mathbb{Z}G \cong \mathbb{Z}G \otimes \mathbb{Z}$. So $\bigoplus_i (\mathbb{Z}G) = \bigoplus_i (\mathbb{Z}G \otimes \mathbb{Z})$
$= \mathbb{Z}G \otimes (\bigoplus_i \mathbb{Z}) = \mathbb{Z}G \otimes A_{\leftarrow \text{abelian}}$.
So by b) $F_G \xrightarrow[\sim]{N} F^G$.

2) $M$ projective $\Rightarrow \exists\ F \cong M \oplus K$.
We have $F_G \cong F \otimes_G \mathbb{Z} \cong (M \oplus K) \otimes_G \mathbb{Z} \cong M_G \oplus K_G$.
And $F^G \cong M^G + K^G$ by inspecting action $g \cdot (m, k) = (gm, gk)$.

Thus $\bar{N} : M_G \oplus K_G \xrightarrow{\sim} M^G \oplus K^G$ iso $(m,k) \longmapsto (Nm, Nk)$.

This restricts to an iso $M_G \oplus \{0\} \xrightarrow{\sim} M^G \oplus \{0\}$ because $\bar{N}$ is $G$-linear in $1^{st}$ coord.