# Random Groups Talk 1 - Rafaela 29/1/2026

Today's Schedule: Motivation
what properties will we care about? → Subgroups → presentations
Models of randomness (in brief)
Gromov's density model

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**Motivation:** The study of random groups can arise by asking the questions:
⇝ What does a "typical" group look like?
⇝ What properties do "typical" elements of a certain group have?

↳ More precisely: What does a finitely presented random group look like?
⇝ For example, is a random finite group presentation hyperbolic / residually finite / does it satisfy a small cancellation condition?
⇝ Given a fixed group on a finite set of generators, we can also ask what a random finitely gen'd subgroup looks like. Is it free / malnormal / finite index?
⇝ We will look at both random presentations and random subgroups.

**Fact (Gromov):** "Almost all groups are hyperbolic"

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**Discrete Representations / A collection of definitions / properties we will care about**
free/reduced, Quasi-convex,

⇝ Discrete Representations: To talk about randomness, we need to use some notion of probability and be able to count discrete representations of f.p. groups / subgroups.

• We care about whether a group/subgroup is free, or a word reduced:
→ We let $F(A)$ be the free group on a non-empty set $A$ of cardinality $>1$
→ Denote $R_n$, $n \in \mathbb{N}$ the set of reduced words of length $n$
$R_{\leq n}$ " " " " at most $n$.

**Defn: Cyclically reduced:** A reduced word $u$ is cyclically reduced if $u^2 = uu$ is also reduced.

$CR_n$ denotes the set of cyclically reduced words of length $n$.

**Defn: Cyclic reduction:** If $u$ reduced, $\exists!\ v, w$ st $w$ is cyclically reduced and $u = v^{-1} w v$

We call $w$ the cyclic reduction of $u$, $\kappa(u)$.

$|R_n| = 2r(2r-1)^{n-1}$, $2r(2r-1)^{n-2}(2r-2) \leq |CR_n| \leq 2r(2r-1)^{n-1}$

## • Subgroups of $F(A)$:

let $\vec{h} = (h_1, \dots, h_k)$, $h_i \in F(A)$.

**Notation:**
- $\langle \vec{h} \rangle$ denotes the subgroup generated by $\vec{h}$: all products of elt's of $\vec{h}$ and their inverses.
- $\langle\langle \vec{h} \rangle\rangle$ denotes the normal closure of $\langle \vec{h} \rangle$: all products of conjugates $h_i^g = g^{-1} h_i g$ of $\vec{h}$, where $1 \leq i \leq k$ and $g \in F(A)$.

$\rightsquigarrow$ The group presented by the relators $\vec{h}$, written $\langle A \mid \vec{h} \rangle$ is the quotient $F(A) / \langle\langle \vec{h} \rangle\rangle$.

## • Quasi-Convexity of Subgroups of $G$:

We care about whether a subgroup of $G$ is free or quasi-convex

**Defn: Geodesic words:** $u \in F(A)$ is geodesic if $u$ has minimum length in $\mu^{-1}(\mu(u))$ whenever $\mu$ is a surjective morphism $\mu: F(A) \to G$.

**Defn: Quasi-Convex:** $H \leq G$ is quasi-convex if $\exists k > 0$ st for every geodesic word $u = a_1 \cdots a_n$ s.t. $\mu(u) \in H$, $\forall\ 1 \leq i < n$, $\exists$ a word $v_i$ of length at most $k$ s.t. $\mu(a_1 \dots a_i v_i) \in H$

**Alternative defn:** A subspace $Y$ of a geodesic metric space $X$ is quasi-convex when $\exists k > 0$ st every geodesic in $X$ that connects a pair of points in $Y$ is contained in the $k$ n-hood of $Y$.

$\rightsquigarrow$ A subgroup of $G$ w/ gen. set $A$ is quasi-convex if it is quasi-convex in $Cay(G, A)$.

Note: Geodesic depends on the choice of gen'ing set, but q-convexity doesn't.

1) **Fact:** Every fin. gen. subgroup of a free group is free and q-convex

## • Malnormality and purity of Subgroups:

malnormal trivial

**Defn:** $H \leq G$ is almost malnormal if $H^g \cap H$ is finite $\forall g \notin H$.

$\Downarrow$

pure $\forall x \in G$ (including finite)

**Defn:** $H \leq G$ is almost pure if $x^n \in H \Rightarrow x \in H$ for any $n \in \mathbb{N}$, $x \in G$ of infinite order

Note: almost = general in torsion-free groups

(almost) malnormal $\Rightarrow$ (almost) pure

2) **Fact:** It is decidable whether a fin. gen. subgroup of a free group is malnormal and whether it is pure.

**Note:** Facts 1, 2 are not decidable for general fin. presented groups. even hyperbolic ones!

**Note:** Almost malnormal is decidable for q-convex subgroups of hyp. groups.

## • Properties of presentations:

Combinatorial — eg small cancellation

Geometric — eg hyperbolicity

### The $C'(\lambda)$ small-cancellation property:

basically, common subword

**Defn:** A piece in a tuple $\vec{h}$ of cyclically reduced words is a word $u$ that occurs as a prefix of two distinct elements of the set of cyclic conjugates of the elements of $\vec{h}$ and their inverses

**Examples:** $\vec{h} = (a^{-1}a^{-1}bb,\ bab^{-1}ab,\ a^{-1}ba^{-1}b)$ inverse $= b^{-1}ab^{-1}a$

So $ab^{-1}a$ is a piece!

$\rightsquigarrow$ A finite presentation $\langle A | \vec{h} \rangle$ satisfies the small cancellation property $C'(\lambda)$ if a piece $u$ in $\vec{h} = (h_1, \dots, h_k)$ satisfies $|u| < \lambda |h_i|$ $\forall i$ st $u$ is a "piece"

"no two relators share a common subword of length more than $\lambda$ times the inf of their lengths"

## • Why would we care about small cancellation?

$C'(\frac{1}{6}) \Rightarrow$ hyperbolicity of the group presented by $\langle A | \vec{h} \rangle$.

♡

We saw some main motivation of studying random groups, and an overview of properties of subgroups and presentations we care about. How likely is it that a group/subgroup/presentation has these properties?

## Randomness

What does the statement "most groups are hyperbolic" actually mean?

→ It is a statistical statement:

Out of all possible presentations, asymptotically most of them define hyperbolic groups

The asymptotics are taken wrt the length of the relators

To give some context to our previous definition of small-cancellation and how it relates to us:

**Few Relator model**: This is a (relatively straightforward) model of randomness:

We let $R_{k,\ell}$ be the set of all possible presentations with $k$ relators of length at most $\ell$.

Let $P$ be a property of a presentation. $P$ occurs with overwhelming probability in this model if the share of presentations in $R_{k,\ell}$ which have property $P$ tends to $1$ as $\ell \to \infty$.

**Prop**: $\forall k, \forall \lambda > 0$, the $C'(\lambda)$ small cancellation property occurs with overwhelming probability in the few relator model.

⇝ $\lambda < 1/6 \Rightarrow$ hyperbolicity, hence hyperbolicity also occurs with overwhelming probability.

**Prop**: Torsion-freeness and cohom. dim. 2 also occur w/ overwhelming probability.

# Gromov's Density model:

**Defn:** Let $F_m$ be the free group on $m \geq 2$ gen^tors, $a_1, \dots, a_m$.
For any $\ell \in \mathbb{Z}$, let $S_\ell \subseteq F_m$ be the set of reduced words of length $\ell$ in these gen^tors. → Bassino: $S \subseteq CR_n$

Let $0 \leq d \leq 1$. A random set of relators at density $d$, at length $\ell$, is a $(2m-1)^{d\ell}$-tuple of elements in $S_\ell$, randomly picked among all elements of $S_\ell$ (uniformly and independently).

Bassino $|CR_n|^{\alpha}$-tuple of elements of $CR_n$ = cyclically reduced words of length $n$.

A random group at density $d$, at length $\ell$ is the group $G = \langle a_1, \dots, a_m, R \rangle$ where $R$ is a random set of relators at density $d$ at length $\ell$.

Notation note: This is Ollivier's notation, which I find more straightforward. Bassino uses our previous definitions nicely (see notes in pink)

→ Bassino: "generic" (opp. negligible)

⇝ A property of $R$ (or of $G$) occurs with overwhelming probability at density $d$ if its probability of occurence tends to 1 as $\ell \to \infty$ for fixed $d$.

⇝ Main point in this is the # of relators taken, which is actually quite large!

**Notes on this definition:**
- $S_\ell$ contains $\sim (2m-1)^{\ell}$ words, so the density is measured logarithmically.
- Basic idea: $d\ell$ is the "dimension" of the random set of relators $R$.
  (here we consider $S_\ell$ to be of dim. $\ell$ because we have $\ell$ letter choices)

**Prop:** $R$ a random set of relators at density $d$, at length $\ell$.
$0 \leq \alpha < d$: with overwhelming probability, any reduced word of length $\alpha\ell$ appears as a subword of some word in $R$

**Intuition:** let's look at the case of a set of $2^{\ell d}$ random words of length $\ell$, in the two letters $a, b$.
We can use a counting argument to show that very probably, one of these words will begin with $\sim d\ell$ "$a$"s (but not much more)

## ⇝ What about small cancellation?

**Prop:** • $\lambda > 0$, $d < \lambda/2$. With overwhelming probability, a random set of relators at density $d$ satisfies the $C'(\lambda)$ small cancellation condition.

• $\lambda > 0$, $d > \lambda/2$. With overwhelming probability, a random set of relators at density $d$ does <u>not</u> satisfy the $C'(\lambda)$ S.C.C.

**Proof: intuition:** What is the probability that $\exists$ two relators in $\mathcal{R}$ sharing a common subword of length $\lambda l$.
The dimension of $\mathcal{R}$ is $dl$ $\Rightarrow$ the dimension of the set of couples $\mathcal{R}\times\mathcal{R}$ is $2dl$.

Note: Classically, the dimension of a set is the number of "independent equations" that we can impose s.t. there is still an element in the set satisfying them. For words, an equation means prescribing some letter into the word.

With this in mind, sharing a subword of length $L$ imposes $L$ many equations, hence the dimension of the set of couples of relators sharing a common subword of length $\lambda l$ is $2dl - \lambda l$

$\to -\infty$ as $l \to \infty$ if $2d < \lambda$

$\to +\infty$ as $l \to \infty$ if $2d > \lambda$

## ⇝ The Phase Transition:

**THM (Gromov): (Phase transition thm):** let $G$ be a random group at density $d$:

• If $d < 1/2$, then with overwhelming probability, $G$ is infinite, hyperbolic, torsion-free, of geometric dimension 2

• If $d > 1/2$, then with overwhelming probability, $G$ is either $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$.

Proof: Potentially final lecture!

## • Some notes on density $1/2$:

→ What happens at $d = 1/2$ is unknown!

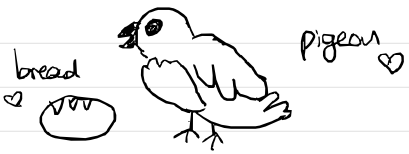
→ For $d < 1/12$ → presentation satisfies the $C'(1/6)$-cc ♡♡ so for $d < 1/12$ the theorem follows!

→ Why density $1/2$???

↳ The probabilistic pigeon-hole principle "birthday paradox" says that

→ in a class of $> 23$ kids, it's very likely that two will share a birthday.

OR → Putting more than $\sqrt{N}$ pigeons in $N$ holes very likely results in two pigeons sharing




This implies that a generic set of density $1/2$ self-intersects, ie we very likely picked the same relator twice.
Eventually, we get that any generator will be equal to every other and its inverse, so $G$ will have one or two elements.

⇝ More on this in the bonus lecture!