Arithmetic Geometry - Notes for Week 2

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We will cover Chapters 2 and 3 (up to Section 3.3) of Poonen's *Rational Points on Varieties* today.

Varieties, base extension, scheme-valued points and curves

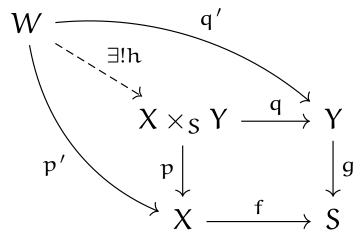
Schemes

This section is a reminder of some relevant definitions.

Definition (*adjective* scheme)**.** A scheme X is:

- *locally noetherian* if X has a cover $\{X_i = \operatorname{Spec} A_i\}$ where each A_i is a noetherian ring;
- *noetherian* if in addition the cover can be taken to be finite;
- *connected* if its underlying topological space is connected;
- *irreducible* if its underlying topological space is irreducible, i.e. whenever $X = X_1 \cup X_2$ with X_1, X_2 closed, either $X_1 = X$ or $X_2 = X$;
- *reduced* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents;
- *integral* if X is non-empty and for every $U \subseteq X$ open and non-empty, $\mathcal{O}_X(U)$ is an integral domain, or equivalently, if X is reduced and irreducible;
- *normal* if all local rings of X are integrally closed domains;
- *regular* if all local rings of X are regular, i.e. $\dim \mathcal{O}_{X,x} = \dim_{K(X)} \mathfrak{m}_x/\mathfrak{m}_x^2$.

Given morphisms of schemes $f : X \to S$ and $g : Y \to S$, the *fibre product* of X and Y is a scheme $X \times_S Y$ together with morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ such that $p \circ f = q \circ g$, and given any scheme T with morphisms $p' : W \to X$ and $q' : W \to Y$, there exists a unique map $h : W \to X \times_S Y$ with $p' = h \circ p$ and $q' = h \circ q$. This is represented in the commutative diagram:



Theorem. *Fibred products exist in the category of schemes.*

Definition (Closed immersion)**.** A morphism of schemes $f : X \to Y$ is *a closed immersion* if f induces a homeomorphism between X and a closed subset of Y, and the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

Definition (*adjective* morphism of schemes)**.** A morphism of schemes $f : X \to Y$ is

- *locally of finite type* if there is a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by open affine schemes $U_{ij} = \operatorname{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra under the ring map $B_i \to A_{ij}$ induced by $f|_{U_{ij}} : U_{ij} \to V_j$;
- *of finite type* if in addition the cover $\{U_{ij}\}$ of $f^{-1}(V_i)$ for each i can be taken to be finite;
- *separated* if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion;
- *universally closed* if for any morphism $Y' \to Y$, the induced morphism $X \times_Y Y' \to Y'$ is a closed morphism taking closed sets to closed sets;
- *proper* if it is separated, of finite type and universally closed;
- *projective* if it factors into a closed immersion $X \to \mathbb{P}^n_Y = \operatorname{Proj} Y[x_0, \dots, x_n]$ for some n, followed by a projection $\mathbb{P}^n_Y \to Y$.

For S a scheme, a *scheme over* S or an S-*scheme* is a scheme X equipped with a morphism $f : X \to S$, called the *structure morphism*. We say X is *adjective* over S if the structure morphism $X \to S$ is *adjective*.

An S-*morphism* between S-schemes X and Y is a morphism of schemes $X \to Y$ such that

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

commutes. Write Scheme_S to denote the category of schemes over S and $\operatorname{Hom}_S(X, Y)$ the set of S-morphisms from X to Y.

Note. For a ring R, we sometimes write R as an abbreviation for $\operatorname{Spec} R$, the use of which is clear from context. For example, $X \times_k Y$ and $X \times_{\operatorname{Spec} k} Y$ are exchangeable.

Varieties

We use a definition of a variety that is more expansive.

Definition (k-variety)**.** A *variety over* k is a separated scheme X of finite type over $\operatorname{Spec} k$.

A *curve* is a variety of pure dimension 1, a *surface* is a variety of pure dimension 2, a 3-*fold* is a variety of pure dimension 3, and so on. The dimension of a variety is its dimension as a topological space (i.e. the length of the longest chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d$ of closed irreducible subsets), and pure means that all irreducible components have the same dimension.

Base change in algebraic geometry

Let S be a fixed scheme which we think of as a *base scheme*, meaning that we are interested in the category of schemes over S.

Definition (Base extension)**.** If S and S' are base schemes and $S' \to S$ is a morphism, then for any scheme X over S, the *base extension* of X is the S'-scheme $X_{S'} = X \times_S S'$. The *base extension* of a morphism of S-schemes $f : X \to Y$ is the induced S'-morphism $f' : X_{S'} \to Y_{S'}$, namely $\mathrm{id}_{S'} \times_{\mathrm{id}_S} f$.

Examples. 1. If X is a k-variety or a k-scheme and $k \subseteq L$ is a field extension, then X_L is the scheme defined by the same equations but instead considered over L.

2. Let X be a k-scheme, and let $\sigma \in \operatorname{Aut} k$. The base extension of X induced by the morphism $\sigma^* : \operatorname{Spec} k \to \operatorname{Spec} k$ is a new k-scheme ${}^\sigma X$.

$$\begin{array}{ccc} {}^\sigma X & \dashrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\sigma^*} & \operatorname{Spec} k \end{array}$$

   X and ${}^\sigma X$ are isomorphic as abstract schemes but not necessarily as k-schemes. For example, if E/k is an elliptic curve and k is an algebraically closed field, then E and ${}^\sigma E$ are isomorphic over k if and only if $j(E) = j({}^\sigma E) = \sigma(j(E))$, where $j(E)$ is the j-invariant of E.

3. Let S be a scheme and let X be an S-scheme with structure morphism $f : X \to S$. If U is an open subscheme of the base scheme S, then X_U is also written $f^{-1}U$ since its underlying topological space is $f^{-1}U$. Ditto for closed subschemes of S.

4. Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$ and let $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ be the residue field of the local ring $\mathcal{O}_{Y,y}$. There is a natural morphism $\operatorname{Spec} \kappa(y) \to Y$. The *scheme-theoretic fibre* at y is $X_y = \operatorname{Spec} \kappa(y) \times_Y X$, alternatively written $f^{-1}(y)$ which is homeomorphic to its underlying topological space.

   If A is a ring, X is an A-scheme and $\mathfrak{p} \subseteq A$ is a prime ideal, then the fibre $X_\mathfrak{p}$ is called the *reduction of* X *modulo* $\mathfrak{p}$.

5. Let X be the affine plane curve over $\mathbb{Q}$ defined by the equation $x^2 + y^2 = 1$, so that $X = \operatorname{Spec} \mathbb{Q}[x,y]/(x^2+y^2-1)$. Let Y be the plane curve defined by $x^2+y^2+1 = 0$. Let $L = \mathbb{Q}[i]$. Then $X_L \simeq Y_L$ as L-varieties, but $X \not\simeq Y$ since $\mathbb{Q}[x,y]/(x^2+y^2-1)$ admits a $\mathbb{Q}$-algebra homomorphism to $\mathbb{Q}$ (i.e. X has a $\mathbb{Q}$-rational point) but $\mathbb{Q}[x,y]/(x^2+y^2+1)$ does not.

A variety may lose integrality, connectedness, irreducibility, reducedness or regularity by base extension of the ground field. Hence we introduce the following definition:

Definition (Geometrically *adjective* scheme)**.** Let X be a scheme over a field k. X is *geometrically integral* if and only if $X_{\bar{k}}$ is integral. Similarly define *geometrically connected, geometrically irreducible*, *geometrically reduced* and *geometrically regular*.

Examples (*adjective* does not imply geometrically *adjective*).

1. Let $k = \mathbb{R}$ and let $X = \operatorname{Spec} \mathbb{R}[x]/(x^2+1)$. Then $X \cong \operatorname{Spec} \mathbb{C}$ is a single point and $X_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x]/(x-i) \times \mathbb{C}[x]/(x+i))$ is the disjoint union of two points. X is connected, irreducible and integral, but it is not geometrically connected, geometrically irreducible or geometrically integral. X is, however, geometrically reduced.

2. Let $k = \mathbb{F}_p(t)$ and let $X = \operatorname{Spec} k[x]/(x^p - t)$. Then $x^p - t$ is irreducible over k and $k[x]/(x^p - t)$ is a field, so X is reduced. $x - t^{1/p}$ is a nilpotent element of the ring $\bar{k}[x]/(x^p - t)$, so X is not geometrically reduced.

3. Let $k = \mathbb{F}_p(t)$ and let X be the curve $y^2 = x^p - t$. Then X is regular since the partial derivatives of $f(x, y) = y^2 - x^p + t$ do not simultaneously vanish at any $x, y \in k$. Now $f(x, y) = y^2 - (x - t^{1/p})$ over $\bar{k}$, so $X_{\bar{k}}$ has a singularity at $(t^{1/p}, 0)$ and X is not geometrically regular.

Function fields

If X is an integral scheme, then there is a unique *generic point* η such that $\overline{\{\eta\}} = X$. The stalk of $\mathcal{O}_X$ at η is a field, called the *function field*, denoted by $K(X)$. If $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the field of fractions of A.

Proposition. *Let* L *be a finitely generated field extension of* k*. Then there exists a normal projective integral* k*-variety* X *with* $K(X) \simeq L$.

For an integral k-variety X, the properties of geometrically irreducible, geometrically reduced and geometrically integral are equivalent to field-theoretic properties of the field extension $K(X)/k$.

Definition (Separable field extension). A field extension L/k is *separable* if the ring $L \otimes_k k'$ is reduced for all field extensions k'/k.

This definition of separable agrees with the usual notion for algebraic field extensions.

Proposition. *Let* L/k *be a finitely generated field extension.*

(i) L *is separable over* k *if and only if* L *is a finite separable extension of a purely transcendental extension* $k(t_1, \dots, t_n)$.

(ii) *Let* $n = \operatorname{tr} \deg(L/k)$. *Elements* $t_1, \dots, t_n$ *of* L *generate a purely transcendental extension of* k *over which* L *is a finite separable extension if and only if* $dt_1, \dots, dt_n$ *form a basis for the* L*-vector space* $\Omega_{L/k}$ *of Kähler differentials.*

Example. If L is separable over k, then every subextension is separable over k, and in particular every finite subextension is separable over k.

Let $k = \mathbb{F}_p(s, t)$ and let L be the function field of the variety X in $\mathbb{A}^2_k$ defined by $sx^p + ty^p = 1$. $\operatorname{tr} \deg(L/k) = 1$ since X is a curve, the only finite subextension of k contained in L is k and L is not separable over k since $L \otimes_k k'$ is not reduced for $k' = \mathbb{F}_p(s^{1/p}, t^{1/p})$.

Definition (Primary field extension)**.** A field extension L/k is *primary* if the largest separable algebraic extension of k contained in L is itself.

Purely inseparable algebraic field extensions are primary. Purely transcendental field extensions are primary and separable.

Proposition. *Let X be a k-scheme of finite type. Then TFAE:*

(i) X *is geometrically irreducible.*

(ii) *There is a separably closed field L containing k such that the L-scheme X_L is irreducible.*

(iii) *For all fields L containing k, the L-scheme X_L is irreducible.*

(iv) X *is irreducible, and the field extension $K(X)/k$ is primary.*

Proposition. *Let X be a k-scheme of finite type. Then TFAE:*

(i) X *is geometrically reduced.*

(ii) *There is a perfect field L containing k such that the L-scheme X_L is reduced.*

(iii) *For all fields L containing k, the L-scheme X_L is reduced.*

(iv) X *is reduced, and for each irreducible component Z of X, the field extension $K(Z)/k$ is separable.*

Combining the two gives conditions for geometric integrality:

Proposition. *Let X be a k-scheme of finite type. Then TFAE:*

(i) X *is geometrically integral.*

(ii) *There is an algebraically closed field L containing k such that the L-scheme X_L is integral.*

(iii) *For all fields L containing k, the L-scheme X_L is integral.*

(iv) X *is integral, and the field extension $K(X)/k$ is primary and separable.*

Let L/k be a finitely generated field extension, so $L = K(X)$ for some integral k-scheme X of finite-type. The *constant field* of X is the maximal algebraic extension k' of k contained inside L. If $t_1, \dots, t_n$ is a transcendence basis of L/k, then

$$[k' : k] = [k'(t_1, \dots, t_n) : k(t_1, \dots, t_n)] \leqslant [L : k(t_1, \dots, t_n)] < \infty,$$

so k' is a finite extension of k.

Proposition. *Let X be an integral k-scheme of finite-type and let $k' \supseteq k$ be its constant field.*

(i) *If X is geometrically integral, then $k' = k$.*

(ii) *If X is proper, then $\mathcal{O}_X(X)$ is a subfield of k'.*

(iii) *If X is normal, then $k' \subseteq \mathcal{O}_X(X)$.*

Scheme-valued points

Let X be a subvariety of $\mathbb{A}^n_k$ defined by the system of polynomial equations

$$\begin{aligned} f_1(x_1, \dots, x_n) &= 0 \\ f_2(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_m(x_1, \dots, x_n) &= 0. \end{aligned}$$

Then $X = \operatorname{Spec} A$ where $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$. A k-*rational point* on X is an n-tuple $(a_1, \dots, a_n) \in k^n$ such that $f(a_1, \dots, a_n) = \cdots = f_m(a_1, \dots, a_n) = 0$. The set of k-rational points on X is in bijection with the set $\operatorname{Hom}_{k\text{-Alg}}(A, k)$ which is in bijection with the set $\operatorname{Hom}_{k\text{-Sch}}(\operatorname{Spec} k, X)$.

Definition (Scheme-valued point)**.** Let X and T be S-schemes. A T-*valued point* of X is an S-morphism $T \to X$. Write $X(T)$ for the set of T-valued points on X.

Notation. If $S = \operatorname{Spec} k$ and $T = \operatorname{Spec} L$ where L/k is a field extension, then an element of $X(L)$ is called an L-*rational point*.

Definition (Functor of points)**.** The *functor of points* of X is the functor

$$\begin{aligned} h_X : \mathsf{Scheme}_S^{op} &\longrightarrow \mathsf{Sets} \\ T &\longmapsto X(T) = \operatorname{Hom}_S(T, X). \end{aligned}$$

In particular, for each S-morphism $f : T' \to T$, then $h_X(f) : X(T) \to (T')$ is the map that sends each S-morphism $\varphi : T \to X$ to the composition $\varphi \circ f : T' \to T \to X$.

A morphism of S-schemes $X \to Y$ induces a map of sets $X(T) \to Y(T)$ for each S-scheme T, and whenever $T' \to T$ is an S-morphism, we obtain a commutative square

$$\begin{array}{ccc} X(T') & \longleftarrow & X(T) \\ \downarrow & & \downarrow \\ Y(T') & \longleftarrow & Y(T) \end{array}$$

In other words, the S-morphism between X and Y induces a *natural transformation of contravariant functors* h_X, h_Y. A corollary of Yoneda's lemma is that:

Lemma. *The set of natural transformations between* h_X *and* h_Y *is exactly* $h_Y(X) = \operatorname{Hom}_S(X, Y)$.

Definition (Fine moduli space)**.** A functor $F : \mathsf{Scheme}_S^{op} \to \mathsf{Sets}$ is *representable* if $F \cong h_M$ for some S-scheme M. We say M *represents* F or M is a *fine moduli space* for F.

Even if F is not representable, it can still be approximated by a functor h_M:

Definition (Coarse moduli space)**.** Let $F : \mathsf{Scheme}_S^{op} \to \mathsf{Sets}$ be a functor. An S-scheme M equipped with a natural transformation $\iota : F \to h_M$ is a *coarse moduli space* for F if:

(i) For every other S-scheme M' with a morphism $F \to h_{M'}$, there is a unique S-morphism $M \to M'$ such that $F \to h_{M'}$ factors as $F \to h_M \to h_{M'}$.

(ii) For every algebraically closed field k and morphism $\operatorname{Spec} k \to S$, ι induces a bijection $F(\operatorname{Spec} k) \to M(k)$.

Fine moduli spaces and coarse moduli spaces are unique, and they are equivalent notions for representable functors.

Example (Moduli space of curves)**.** Fix k to be an algebraically closed field. For $g \in \mathbb{N}$, let $\mathcal{M}_g(k)$ denote the set of smooth projective geometrically integral curves of genus g up to isomorphism. More generally for any scheme T, let $\mathcal{M}_g(T)$ be the set of isomorphism classes of smooth proper T-schemes whose fibres are geometrically integral curves of genus g. The functor $F = \mathcal{M}_g$ is not represented by a scheme since there are nontrivial families of curves all of whose fibres are isomorphic to each other.

In the case $g = 1$, $\mathcal{M}_g(k)$ is parametrized by the j-invariants, which form an affine line $M_1 = \mathbb{A}^1_k$ that acts as a coarse moduli space for $\mathcal{M}_1$. For $g \geqslant 2$, Mumford has shown that there is a coarse moduli space M_g such that:

(i) The set of closed points of M_g is in one-to-one correspondence with the set of isomorphism classes of curves of genus g.

(ii) If $f : X \to T$ is any flat family of curves of genus g, then there is a morphism $h : T \to M_g$ such that for each closed point $t \in T$, X_t is in the isomorphism class of curves determined by the point $h(t) \in M_g$.

For $g \geqslant 2$, M_g is an irreducible quasi-projective variety of dimension $3g-3$ over any fixed algebraically closed field.

Definition (Dominant morphism)**.** A morphism of schemes $f : X \to Y$ is *dominant* if $f(X)$ is dense in the topological space Y. f is *scheme-theoretically dominant* if either of the following equivalent conditions holds:

(i) Whenever U is an open subscheme of Y, and $f|_{f^{-1}U} : f^{-1}U \to U$ factors as $f^{-1}U \to Z \hookrightarrow U$ for some closed subscheme Z of U, we have $Z = U$.

(ii) The sheaf homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.

Scheme-theoretically dominant implies dominant.

Proposition. *Let X be a separated S-scheme. If $T' \to T$ is a scheme-theoretically dominant S-morphism, then $X(T) \to X(T')$ is injective.*

Corollary. *If $R \subseteq R'$ is an inclusion of rings and X is a separated R-scheme, then $X(R) \to X(R')$ is injective.*

Corollary. *Let X be a reduced S-scheme, and let Y be a separated S-scheme. Let U be a dense open subscheme of X. If f and g are morphisms $X \to Y$ such that $f|_U = g|U$, then $f = g$.*

Proposition. *Let k be a field. If X and Y are connected k-schemes and X has a k-rational point, then $X \times_k Y$ is connected. In particular, a connected k-scheme with a k-rational point is geometrically connected.*

Proposition. *Let X be a finite-type scheme over a field k such that $X(k)$ is dense in X. Then:*

(i) If X is irreducible, then X is geometrically irreducible.

(ii) If X is reduced, then X is geometrically reduced.

(iii) If X is integral, then X is geometrically integral.

Closed points

Definition (Closed point)**.** A *closed point* of a scheme X is a point $x \in X$ such that $\{x\}$ is Zariski closed in X.

If X is a variety over an algebraically closed field k, the map $X(k) \to \{\text{closed points in } X\}$ with $(f : \operatorname{Spec} k \to X) \mapsto f(\operatorname{Spec} k)$ is a bijection. The nonclosed points of X are generic points of the positive-dimensional integral subvarieties of X.

Proposition. *Let X be a k-variety and let $x \in X$. Then the following are equivalent:*

(i) The point x is closed.

(ii) The dimension of the closure of $\{x\}$ is 0.

(iii) The residue field $\kappa(x)$ is a finite extension of k.

Definition (Degree of closed point)**.** The *degree* of a closed point x on a k-variety X is the degree of the field extension $\kappa(x)/k$.

Proposition. *Let X be a k-variety. Then the map*

$$\begin{aligned} \{\mathfrak{G}_k\text{-orbits in } X(\bar{k})\} &\longrightarrow \{\text{closed points of } X\} \\ \text{orbit of } (f : \operatorname{Spec} \bar{k} \to X) &\longmapsto f(\operatorname{Spec} \bar{k}) \end{aligned}$$

is a bijection.

In particular, if X is a k-variety, then k-points of X are in bijection with closed points with residue field k.

Curves

Let X be a regular, projective, geometrically integral curve over a field k. The *arithmetic genus* of X is $p_a(X) = h^1(X, \mathcal{O}_X)$ and the *geometric genus* of X is $p_g(X) = h^0(X, \omega_X^\circ)$, where ω_X° is the dualizing sheaf on X. By Serre duality, $H^1(X, \mathcal{O}_X) \simeq H^0(X, \omega_X^\circ)^\vee$, so these are equivalent notions. The *genus* of X is $g(X) = p_a(X) = p_g(X)$.

If Y is a curve birational to a regular, projective, geometrically integral curve X, define $g(Y) = g(X)$.

Theorem. *Let X be a regular, projective, geometrically integral k-curve and let L/k be a field extension. Then:*

(i) We have $g(X_L) \leqslant g(X)$, with equality if and only if X_L is regular.

(ii) The difference $g(X_L) - g(X)$ is divisible by $(p-1)/2$, where $p = \operatorname{char} k > 0$.

(iii) If L is separable over k, then $g(X_L) = g(X)$.

Recall that a *prime divisor* on X is an integral closed subscheme of codimension 1 and the group $\operatorname{Div} X$ of *Weil divisors* is the free abelian group generated by prime divisors. Since X is a curve, the prime divisors of X are the closed points P of X.

Each $D \in \operatorname{Div} X$ gives rise to a line bundle $\mathcal{O}(D)$. This induces an isomorphism from the group of Weil divisors modulo linear equivalence to the *Picard group* $\operatorname{Pic} X$ of isomorphism classes of line bundles.

The *degree* of a divisor $D = \sum n_P P \in \operatorname{Div}(X)$ is $\deg D = \sum n_P \deg P$. The *Riemann-Roch space* of a divisor D is $\mathcal{L}(D) = H^0(X, \mathcal{O}(D))$. Define $l(D) = \dim_k \mathcal{L}(D) = h^0(X, \mathcal{O}(D))$. A *canonical divisor* of X is a divisor K such that $\omega_X^\circ = \mathcal{O}(K)$.

Theorem (Riemann-Roch theorem)**.** *Let X be a regular, projective, geometrically integral k-curve and let K be a canonical divisor of X. Then* $l(D) - l(K - D) = \deg D + 1 - g$.

Rational points over topological fields

If k is a finite field and X is a k-variety, then $X(k)$ is finite. Now let k be a topological field (e.g. a local field) and let X be a k-variety. Define the *analytic topology* on $X(k)$ as follows:

- Give the set $\mathbb{A}_k^n = k \times \cdots \times k$ the product topology.
- If X is a closed subvariety of $\mathbb{A}_k^n$, then give $X(k) \subseteq \mathbb{A}_k^n$ the subspace topology.
- If X is obtained by gluing open affine sets $X_1, \ldots, X_m$, then use the same gluing data to glue the topological spaces $X_1(k), \ldots, X_m(k)$.

Two different affine open coverings give the same topology on $X(k)$. Any morphism of k-varieties $X \to Y$ induces a continuous map $X(k) \to Y(k)$.

Proposition. *Let k be a local field. If $X \to Y$ is a proper morphism of k-varieties, then $X(k) \to Y(k)$ is a proper map of topological spaces, i.e. the inverse image of any compact subset of $Y(k)$ is compact. In particular, if X is a k-variety and X is proper over k, then $X(k)$ is compact. The converse holds when $k = \mathbb{C}$.*

If $k = \mathbb{C}$, then we can equip the topological space $X(\mathbb{C})$ with a sheaf of germs of holomorphic functions to get a locally ringed space X^{an}.

Finiteness conditions, spreading out and flat morphisms

Quasi-compact and quasi-separated morphisms

Definition (Quasi-compact scheme)**.** A scheme X is *quasi-compact* if one of the following equivalent conditions is satisfied:

(i) The topological space of X is quasi-compact, i.e. every open cover of X has a finite subcover.

(ii) The scheme X is a finite union of affine open subsets.

Definition (Quasi-compact morphism)**.** A morphism of schemes $f : X \to Y$ is *quasi-compact* if one of the following equivalent conditions is satisfied:

(i) There is an affine open covering $\{Y_i\}$ of Y such that for each i, the scheme $f^{-1}Y_i$ is quasi-compact.

(ii) For every affine open subset $U \subseteq Y$, the scheme $f^{-1}U$ is quasi-compact.

Definition (Quasi-separated morphism)**.** A morphism of schemes $f : X \to Y$ is *quasi-separated* if one of the following equivalent conditions is satisfied:

(i) There is an affine open covering $\{Y_i\}$ of Y such that whenever X_1, X_2 are affine open subsets of $f^{-1}Y_i$, the intersection $X_1 \cap X_2$ is a union of finitely many affine open subsets.

(ii) For every affine open $U \subseteq S$ and affine open subsets $X_1, X_2 \subseteq f^{-1}U$, the intersection $X_1 \cap X_2$ is a union of finitely many affine open subsets.

(iii) The diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact.

If X is noetherian, then every open subscheme of X is quasi-compact, so every morphism $X \to Y$ is both quasi-compact and quasi-separated.

Example. Let A be a polynomial ring $k[x_1, x_2, \dots]$ in countably many indeterminates over some field k. Let $P \in \operatorname{Spec} A$ be the closed point corresponding to the maximal ideal $(x_1, x_2, \dots)$. Let U be the open subscheme of $\operatorname{Spec} A$ obtained by removing P. Then the open subsets $D(x_i)$ of $\operatorname{Spec} A$ form an open cover for U with no finite subcover, so U is not quasi-compact.

Let X be the infinite-dimensional affine space with a doubled origin, i.e. the scheme obtained by gluing two copies X_1, X_2 of $\operatorname{Spec} A$ along the copy of U in each. The identity morphisms $X_i \to \operatorname{Spec} A$ glue to give a morphism $X \to \operatorname{Spec} A$ that is not quasi-separated, since X_1 and X_2 are affine open subsets whose intersection U is not quasi-compact.

Finite presentation

Definition (Finitely presented algebra)**.** Let A be a commutative ring and let B be an A-algebra, i.e. there is a ring homomorphism $B \to A$. B is a *finitely presented* A*-algebra* if B is isomorphic as an A-algebra to $A[t_1, \ldots, t_n]/I$ for some $n \in \mathbb{N}$ and some finitely generated ideal I of the polynomial ring $A[t_1, \ldots, t_n]$.

Proposition. *Let A be a commutative ring. An A-algebra is finitely generated if it is finitely presented. The converse holds if A is noetherian.*

Example (Finitely generated does not imply finitely presented)**.** Let k be a field and let $A = k[x_1, x_2, \ldots]$. Then the ideal $I = (x_1, x_2, \ldots) \subseteq A$ is not finitely generated, and the finitely generated A-algebra A/I is not finitely presented.

Definition (Morphism locally of finite presentation)**.** Let X be an S-scheme with structure morphism f. Let $x \in X$ and let $s = f(x)$. f is *locally of finite presentation* at x if there exist affine neighborhoods $V = \operatorname{Spec} A$ of s and $U = \operatorname{Spec} B$ of x such that B is of finite presentation over A. f is *locally of finite presentation* if it is locally of finite presentation at every $x \in X$.

Remark. An S-scheme X is locally of finite presentation if and only if for every filtered inverse system of affine S-schemes $\operatorname{Spec} A_i$ (a morphism $f : X \to S$ is affine if $f^{-1}S_0$ is affine for each affine open subscheme S_0 of S), the natural map $\varinjlim X(A_i) \to X(\varinjlim A_i)$ is a bijection.

Definition (Morphism of finite presentation)**.** A morphism $f : X \to S$ is *of finite presentation* if it is locally of finite presentation, quasi-separated and quasi-compact.

In other words, $f : X \to Y$ is a morphism of finite presentation if for every affine open cover $V_j = \operatorname{Spec} A_j$ of Y, each $f^{-1}V_j$ has a finite cover of affine opens $U_{ij} = \operatorname{Spec} B_{ij}$ such that:

- Each B_{ij} is a finitely presented A_j-algebra, i.e. $B_{ij} \simeq A_j[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some m, n and polynomials $f_1, \ldots, f_m$.
- For any two such affine opens U_{ij} and U_{kj}, their intersection $U_{ij} \cap U_{kj}$ is a finite union of affine opens each of which the ring of sections is a finitely generated algebra over A_j.

If Y is locally noetherian, then a morphism $f : X \to Y$ is locally of finite presentation if and only if it is locally of finite type, and it is of finite presentation if and only if it is of finite type.

Spreading out

The principle of *spreading out* is that for schemes of finite presentation, whatever happens over the generic point also happens over some open neighborhood of the generic point.

Theorem (Spreading out). *Let S be an integral scheme with function field $K = K(S)$.*

(i) *Suppose that X is a scheme of finite presentation over K. Then there exist a dense open subscheme $U \subseteq S$ and a scheme $\mathcal{X}$ of finite presentation over U such that $\mathcal{X}_K \simeq X$.*

(Spreading out schemes)

(ii) *Suppose that $\mathcal{X} \to S$ is of finite presentation. If* adjective *is a property that can be spread out and $\mathcal{X}_K \to \operatorname{Spec} K$ is* adjective*, then there exists a dense open subscheme $U \subseteq S$ such that $\mathcal{X}_U \to U$ is* adjective*.* (Spreading out properties of schemes)

(iii) *Suppose that $\mathcal{X}$ and $\mathcal{X}'$ are schemes of finite presentation over S, and $f : \mathcal{X}_K \to \mathcal{X}'_K$ is a K-morphism. Then there exists a dense open subscheme $U \subseteq S$ such that f extends to a U-morphism $\mathcal{X}_U \to \mathcal{X}'_U$.* (Spreading out morphisms)

(iv) *Let $f : \mathcal{X} \to \mathcal{X}'$ be an S-morphism between schemes of finite presentation over S. If* adjective *is a property that can be spread out and $f : \mathcal{X}_K \to \mathcal{X}'_K$ is* adjective*, then there exists a dense open subscheme $U \subseteq S$ such that $f|_U : \mathcal{X}_U \to \mathcal{X}'_U$ is* adjective*.*

(Spreading out properties of morphisms)

The list of properties that can be spread out from a generic fibre to a dense open subscheme as in (ii) *and* (iv) *are: affine, closed immersion, finite, flat, geometrically connected, geometrically integral, geometrically irreducible, geometrically reduced, immersion, isomorphism, étale, smooth, unramified, G-unramified, fppf, fpqc, monomorphism, open immersion, projective, quasi-projective, proper, quasi-affine, quasi-finite, radicial, separated and surjective.*

Remark (Spreading out to an open neighborhood of a point). (i) can be generalized as follows: Let S be a scheme and let $s \in S$. Then a scheme of finite presentation over $\operatorname{Spec} \mathcal{O}_{S,s}$ can be spread out to a scheme $\mathcal{X}$ of finite presentation over some open neighborhood of s in S.

The ring $\mathcal{O}_{S,s}$ is the injective limit of coordinate rings of the affine open neighborhood of s in S, so $\operatorname{Spec} \mathcal{O}_{S,s}$ is a projective limit of schemes.

We give some standard applications of spreading out:

Proposition. *Suppose that X is of finite presentation over a commutative ring A. Then there exists a noetherian ring A_0 contained in A and a scheme X_0 of finite presentation over A_0 with $(X_0)_A \simeq X$.*

If X and Y are $\mathbb{Q}$-varieties whose base extensions $X_{\mathbb{Q}(t)}$ and $Y_{\mathbb{Q}(t)}$ are isomorphic, where t is an indeterminate, then one can specialize t to some rational number q to obtain an isomorphism $X \to Y$.

Proposition (Specializing an isomorphism). *Let L/k be a field extension. If X and Y are k-varieties such that $X_L \simeq Y_L$, then $X_F \simeq Y_F$ for some finite extension F/k.*

Models over discrete valuation rings

Let R be a discrete valuation ring (dvr) with fraction field K, residue field k and uniformizer π, and let X be a proper K-variety. The goal is to make sense of the reduction of X modulo π. The scheme $\operatorname{Spec} R$ consists of two points: the *generic point* $\eta = \operatorname{Spec} K$ corresponding to the prime (0) of R, and the *closed point* $s = \operatorname{Spec} k$ corresponding to the maximal ideal (π) of R.

Definition (Generic fibre and special fibre)**.** Let X_R be an R-scheme. The *generic fibre* of X_R is the fibre above the generic point, i.e. the K-scheme $X_K = X_R \times_{\operatorname{Spec} R} \operatorname{Spec} K$, and the *special fibre* of X_R is the fibre above the closed point, i.e. the k-scheme $X_k = X_R \times_{\operatorname{Spec} R} \operatorname{Spec} k$.

Definition (R-model)**.** Let X be a K-scheme. An R-*model* of X is an R-scheme X_R equipped with an isomorphism $X_R \times_R K \to X$ of K-schemes.

Example. Let $X = \operatorname{Proj} \mathbb{Q}_7[x, y, z]/(xy - 7z^2)$ be a curve over $\mathbb{Q}_7$. Then the schemes $\operatorname{Proj} \mathbb{Z}_7[x, y, z]/(xy - 7z^2)$ and $\operatorname{Proj} \mathbb{Z}_7[x, y, z]/(xy - z^2)$, equipped with suitable isomorphisms, are $\mathbb{Z}_7$ models of X. The special fibre of the former is a reducible curve consisting of the two lines $x = 0$ and $y = 0$, but the special fibre of the latter is irreducible, so they are not isomorphic.

Dedekind domains

We generalize the definition of an R-model to Dedekind domains (i.e. integrally closed noetherian domains of dimension at most 1).

Example.

1. The ring of integers in a number field is a Dedekind domain.
2. The coordinate ring of an affine regular integral curve over a field is a Dedekind domain.
3. Any PID (and hence dvr) is a Dedekind domain.
4. The localization of a Dedekind domain at a prime ideal is a dvr.

A scheme over a Dedekind domain R has one generic fibre and many closed fibres, one for each nonzero prime of R.

Theorem (Valuative criterion for properness)**.** *Let $f : X \to S$ be a morphism of finite type with S Noetherian. Then f is proper if and only if whenever $\operatorname{Spec} R$ is an S-scheme with R a dvr and K its fraction field, the natural map $X(R) \to X(K)$ is bijective, or in other words, whenever given a commutative diagram and for $T = \operatorname{Spec} R$ and $U = \operatorname{Spec} K$, given the a diagram*

$$\begin{array}{ccc} \operatorname{Spec} K = U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} R = T & \longrightarrow & Y \end{array}$$

then there exists a unique map $T \to X$ making the diagram commute.

We generalize the $S = \operatorname{Spec} R$ case to Dedekind domains:

Theorem. *Let R be an integral domain, and let $K = \operatorname{Frac}(R)$. Let X be an R-scheme.*

(i) If X is separated over R, then $X(R) \to X(K)$ is injective.

(ii) If X is proper over R and R is a Dedekind domain, then $X(R) \to X(K)$ is bijective.

Flat morphisms

Recall the algebraic notion of a flat module:

Definition (Flat module)**.** Let A be a ring and let M be an A-module. M is *flat* over A if the functor $N \mapsto N \otimes_A M$ is an exact functor for $N \in A\text{-}\mathsf{Mod}$, i.e. whenever

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence of A-modules, the induced sequence

$$0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$$

is exact. If $A \to B$ is a ring homomorphism, B is *flat* over A if it is flat as a module.

Examples.

1. Free modules are flat. Any module over a field k is flat.
2. A module over a dvr or a Dedekind domain is flat if and only if it is torsion-free.
3. Any localization $S^{-1}A$ of A is flat.

Proposition. *Let A be a ring and let M be an A-module.*

(i) An A-module M is flat if and only if for every finitely generated ideal $\mathfrak{a} \subseteq A$, the map $\mathfrak{a} \otimes M \to M$ is injective.

(ii) If M is a flat A-module and $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module. (Base extension)

(iii) If B is a flat A-algebra and N is a flat B-module, then N is also flat as an A-module. (Transitivity)

(iv) M is flat over A if and only if $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. (Localization)

Definition (Flat morphism)**.** A morphism of schemes $f : X \to Y$ is *flat* at $x \in X$ if $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(y)}$-module. f is *flat* if it is flat at every $x \in X$.

Definition (Faithfully flat morphism)**.** A morphism of schemes $f : X \to Y$ is *faithfully flat* if it is flat and surjective.

If $A \to B$ is a homomorphism of commutative rings, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if B is flat over A, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat if and only if B is flat over A and $M \otimes_A B \neq 0$ for every non-zero A-module M.

Dimension and relative dimension

Definition (Dimension)**.** Let X be a topological space. The *dimension* $\dim X$ of X is the element of $\{-\infty, 0, 1, 2, \ldots, \infty\}$ defined by the formula

$$\dim X = \sup\{n \in \mathbb{N} : X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \text{ irreducible closed subsets of } X\}.$$

The *dimension* of X at $x \in X$ is

$$\dim_x X = \inf\{\dim U : x \in U \subseteq X \text{ open}\}.$$

Remark. The empty set is not irreducible. $\dim X = -\infty$ if and only if $X = \varnothing$.

Theorem. *Let X be a scheme locally of finite type over a field k and let $x \in X$. Then*

$$\dim_x X = \dim \mathcal{O}_{X,x} + \operatorname{tr deg}(\kappa(x)/k).$$

Definition (Relative dimension)**.** Let $f : X \to S$ be a continuous map of topological spaces and let $x \in X$. The *relative dimension* of X over S at x is

$$\dim_x f = \dim_x f^{-1}(f(x)).$$

Proposition. *Let $f : X \to S$ be a flat k-morphism between irreducible k-varieties. Then*

$$\dim_x f = \dim X - \dim S.$$

In particular, $\dim_x f$ *is independent of* x*.*