K-Theory Lecture 1

Idea: associate 2 abelian groups, K_0 and K_1 to each C^*-algebra. This tells us about the structure of the C^*algebra, and allows us to distinguish C^*-algebras by distinguishing their K-theories

1. C^*-Algebra Theory

DEF.
A C^*-Algebra A is an algebra over $\mathbb{C}$ with a norm $\|\cdot\|$ and an involution $*$ s.t. A is complete w.r.t. $\|\cdot\|$ and

$$\|ab\| \le \|a\|\|b\| \quad \forall a,b \in A$$

and

$$\|a^*a\| = \|a\|^2 \quad \forall a \in A. \quad (C^*\text{-condition})$$

DO EXAMPLES

Terminology:

- A is unital if it has a multiplicative identity
- A $*$-homomorphism $\varphi: A \to B$ is a linear, multiplicative map with $\varphi(a^*) = \varphi(a)^*$ $\forall a \in A$.
  If A, B unital and $\varphi(1_A) = 1_B$ then φ is unital/unit preserving.
- A is separable if it has a countable dense subset.
- Sub-$*$-algebra and sub-C^*-algebra as you would expect (C^* requires norm-closure)
- If $F \subseteq A$ then $C^*(F)$ is the sub-C^*-algebra generated by F.

THM (Gelfand-Naimark)
For every C^*-algebra A there exist a Hilbert space H and an isometric $*$-homomorphism $\varphi: A \hookrightarrow \mathbb{B}(H)$. So every C^*-algebra is isomorphic to a sub-C^*-algebra of some $\mathbb{B}(H)$. If A is separable then H can be chosen to be separable.

Ideals and quotients work as usual.
Note:

- Ideals are sub-C^*-algebras
- 'Ideal' means closed, 2-sided

Let $\varphi: A \to B$ be a $*$-homom. Then
(i) $\|\varphi(a)\| \leq \|a\|$ $\forall a \in A$
(ii) φ is injective $\Leftrightarrow$ φ is isometric
(iii) $\ker(\varphi)$ is an ideal in A
(iv) $\operatorname{im}(\varphi)$ is a sub-C^*-alg. of B.

A is simple if the only ideals are 0 and A.

Short exact sequences

A (finite/infinite) sequence of C^*-algebras and $*$-homom.s

$$\cdots \longrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} A_{n+2} \longrightarrow \cdots$$

is exact if $\operatorname{im}(\varphi_n) = \ker(\varphi_{n+1})$ $\forall n$.
An exact sequence

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \quad (\star)$$

is called short exact.
If I is an ideal in A then

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

is a S.E.S.. Conversely, in $(\star)$, $\varphi(I)$ is an ideal in A, $B \cong A/\varphi(I)$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\varphi} & A & \xrightarrow{\psi} & B & \longrightarrow & 0 \\ & & \varphi\downarrow\cong & & \| & & \downarrow\cong & & \\ 0 & \longrightarrow & \varphi(I) & \xrightarrow[\iota]{} & A & \xrightarrow[\pi]{} & A/\varphi(I) & \longrightarrow & 0. \end{array}$$

In $(\star)$ if there is a $*$-homom. $\lambda: B \to A$ s.t. $\psi \circ \lambda = id_B$ then λ is called a lift of ψ and $(\star)$ split exact.

The direct sum $A \oplus B$ of 2 C^*-algebras is given the norm

$$\|(a,b)\| = \max\{\|a\|, \|b\|\}$$

Adjoining a Unit

To every C^*-alg. A, we can associate a unique unital C^*-algebra $\tilde{A}$ with A as an ideal and $\tilde{A}/A \cong \mathbb{C}$. $\tilde{A}$ is called the unitisation of A.

$$\tilde{A} = \{a + \alpha 1_{\tilde{A}} : \alpha \in \mathbb{C}, a \in A\}.$$

Adjoining a unit is functorial. DO THIS

If $A \subseteq B$ with B unital ~~then~~ and $1_B \notin A$, then $A \cong A + \mathbb{C} \cdot 1_B$.

If A is unital then $f = 1_{\tilde{A}} - 1_A$ is a projection in $\tilde{A}$ and

$$\tilde{A} = \{a + \alpha f : a \in A, \alpha \in \mathbb{C}\} \cong A \oplus \mathbb{C}.$$

Spectral Theory

Let A be unital and $a \in A$. The spectrum of a, $sp(a)$ is $\{\lambda \in \mathbb{C} : a - \lambda \cdot 1 \text{ not invertible}\}$.

The spectral radius $r(a)$ of a is

$$r(a) = \sup\{|\lambda| : \lambda \in sp(a)\}.$$

- $sp(a)$ is a closed subset of $\mathbb{C}$ }
- $r(a) \leq \|a\|$ } $\Rightarrow sp(a)$ is cpct.
- $sp(a) \neq \emptyset$
- $r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$

If A is not unital, embed A in $\tilde{A}$ and let $sp(a)$ be $sp(a)$ in $\tilde{A}$. In this case $0 \in sp(a)$ $\forall a \in A$.

More terminology:

$a \in A$ is

- self-adjoint if $a = a^*$
- normal if $aa^* = a^*a$
- positive if normal and $sp(a) \subseteq \mathbb{R}^+ \cup \{0\}$

- the set of positive elements is denoted A^+.
- every self-adjoint element has $sp(a) \subseteq \mathbb{R}$
- a is positive iff $a = x^*x$ for some $x \in A$
- if a is normal, $r(a) = \|a\|$.

States

A linear map $\rho : A \to \mathbb{C}$ is called a linear functional.

The operator norm of ρ is

$$\|\rho\| = \sup\{|\rho(a)| : a \in A, \|a\| \leq 1\}.$$

- ρ is cts. iff $\|\rho\| < \infty$.
- If $\rho(a) \geq 0$ for every $a \in A^+$ then ρ is positive

A <u>state</u> ρ on a unital C^*-alg. A is a positive lin. functional with $\rho(1)=1$ (equiv. $\|\rho\|=1$). The set of states on a unital C^*-alg. separates points (if $\rho(a)=0$ for every state ρ then $a=0$)

<u>THM</u> (Gelfand)

Every abelian C^*-alg. is isometrically $*$-isomorphic to the C^*-alg. $C_0(X)$ for some locally cpct. Hausdorff space X.

($C_0(X)$ is the C^*-alg. of cts. functions $f: X \to \mathbb{C}$ vanishing at ∞ with sup. norm. If X cpct. then $C_0(X) = C(X)$).

In addition:

(i) $C_0(X)$ unital iff X cpct.

(ii) $C_0(X)$ separable iff X separable

(iii) $X \cong Y$ iff $C_0(X) \cong C_0(Y)$

(iv) cts. $g: Y \to X$ induces a $*$-homom. $\varphi: C_0(X) \to C_0(Y)$ by $\varphi(f) = f \circ g$. Conversely for every $*$-homom. $\varphi: C_0(X) \to C_0(Y)$ there is a cts. function $\hat{\varphi}: \cancel{X \to X}$ s.t. $\varphi(f) = f \circ \hat{\varphi}$.

(v) There is a bijective corr. between open subsets of X and ideals in $C_0(X)$. The ideal corresponding to $U \subset X$ is $\{f \in C_0(X) : f(U^c) = 0\}$ and is isomorphic to $C_0(U)$.
The $*$-homom. $C_0(X) \to C_0(U^c)$ given by $f \mapsto f|_{U^c}$ is surjective.

Continuous Functional Calculus

Let A be a unital C^*-alg.. For any normal $a \in A$ there is exactly 1 $*$-isom.

$$C(sp(a)) \to C^*(a,1) \subseteq A$$
$$f \mapsto f(a)$$

s.t. $\iota(a) = a$ (where $\iota \in C(sp(a))$ is $\iota(z) = z \ \forall z$).
If f is a polynomial then $f(a)$ agrees with the usual definition (i.e. $x^n \mapsto a^n$ etc.). Also $\tau(a) = a^*$ where $\tau : \mathbb{C} \to \mathbb{C}$ is $\tau(z) = \bar{z}$.

Spectral Mapping Theorem

If a is normal and f is a cts. function on $sp(a)$ then $sp(f(a)) = f(sp(a))$.

If $\varphi : A \to B$ is a unital $*$-homom. and $a \in A$ is normal, then $sp(\varphi(a)) \subseteq sp(a)$ and $f(\varphi(a)) = \varphi(f(a)) \ \forall f \in C(sp(a))$.

If A is non-unital, $f(a)$ is defined to be an element of $\tilde{A}$. In this case $f(a) \in A \iff f(0) = 0$.

LEM 1.2.5

Let $K \subset \mathbb{R}$ be non-empty and cpct., and $f : K \to \mathbb{C}$ be cts.. Let A be a unital C^*-alg. and Ω_K be the self-adjoint elements of A with spectrum contained in K. Then the induced function $f : \Omega_K \to A$ given by

$$a \mapsto f(a)$$

is cts.

Proof

Since multiplication is cts., $a \mapsto a^n$ is cts. $\forall n \in \mathbb{Z}_{\geq 0}$. So every polynomial induces a cts. map $A \to A$ given by $a \mapsto f(a)$.

Let $f: K \to \mathbb{C}$ be cts., $a \in \Omega_K$ and $\varepsilon > 0$. By the Stone-Weierstrass theorem there is a polynomial g s.t. $|f(z) - g(z)| \leq \varepsilon/3 \ \forall z \in K$. Find $\delta > 0$ s.t. $\|g(a) - g(b)\| \leq \varepsilon/3 \ \forall b \in A$ with $\|a - b\| \leq \delta$. Since $\|f(c) - g(c)\| = \|(f-g)(c)\| = \sup\{|(f-g)(z)| : z \in sp(c)\} \leq \varepsilon/3$ $\forall c \in \Omega_K$, we have $\|f(a) - f(b)\| \leq \varepsilon \ \forall b \in \Omega_K$ with $\|a - b\| \leq \delta$.

■

This can also be shown for any non-empty K, and for $\Omega_K = \{a \in A : a \text{ normal and } sp(a) \subseteq K\}$.

Matrix Algebras

Let A be a C^*-alg. and $n \in \mathbb{N}$. $M_n(A)$ is the set of $n \times n$ matrices $(a_{ij})_{i,j}$ with $a_{ij} \in A$.

Equip $M_n(A)$ with the usual component-wise addition and scalar multiplication, matrix multiplication and define $(a_{ij})_{i,j}^* = (a_{ji}^*)_{i,j}$.

To define a C^*-norm on $M_n(A)$, choose a Hilbert space H and an injective $*$-homom. $\varphi: A \to B(H)$. Let $\varphi_n : M_n(A) \to B(H^n)$ be given by

$$\varphi_n (a_{ij})_{i,j} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = (\varphi(a_{ij}))_{i,j} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \qquad \xi_j \in H.$$

Define a norm on $M_n(A)$ by $\|a\| = \|\varphi_n(a)\|$.

This makes $M_n(A)$ a C^*-alg., and the norm is independent of φ.

Also

$$\max_{i,j} \{\|a_{ij}\|\} \leq \|(a_{ij})_{i,j}\| \leq \sum_{i,j} \|a_{ij}\|.$$

This means that $f : X \to M_n(A)$ is cts. iff each $f_{ij} : X \to A$ is cts. ▪

Forming matrix algebras is functorial: If A and B are C^*-alg.s and $\varphi : A \to B$ is a $*$-homom. then $\varphi_n : M_n(A) \to M_n(B)$ given by

$$\varphi_n((a_{ij})_{i,j}) = (\varphi(a_{ij}))_{i,j}$$

is a $*$-homom. for every n. We will often write φ instead of φ_n.

Examples of C^*-Algebras

- The algebra of all bounded operators on a Hilbert space H, $B(H)$ with the usual adjoint operation:

$$\|a^* a\| = \sup_{\|x\|=\|y\|=1} |\langle a^* a x, y\rangle| = \sup_{\|x\|=\|y\|=1} |\langle ax, ay\rangle| = \|a\|^2$$

- The algebra $C_0(X)$ of al cts. functions on a locally cpct. Hausdorff space X vanishing at infinity. The adjoint operation is complex conjugation and the norm is the sup. norm:

$$\|\bar{f} f\| = \sup_{x \in X} |\overline{f(x)}\, f(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|^2.$$

Terminology

The adjoint (involution) satisfies

(i) $(a+b)^* = a^* + b^*$

(ii) $(\lambda a)^* = \bar{\lambda} a^*$

(iii) $a^{**} = a$

(iv) $(ab)^* = b^* a^*$

Hilbert space - complete normed space where the norm comes from an inner product.

Inner product satisfies:

(i) $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0 \Rightarrow x = 0$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$

Adjoining a unit is functorial

If $\varphi : A \to B$ is a $*$-homom, then there is exactly 1 $*$-homom.
$\tilde{\varphi} : \tilde{A} \to \tilde{B}$ making

$$\begin{array}{ccccccccc} 0 & \to & A & \to & \tilde{A} & \to & \mathbb{C} & \to & 0 \\ & & \varphi\downarrow & & \tilde{\varphi}\downarrow & & \| & & \\ 0 & \to & B & \to & \tilde{B} & \to & \mathbb{C} & \to & 0 \end{array}$$

commute.

$\tilde{\varphi}$ is given by $\tilde{\varphi}(a + \alpha 1_A) = \varphi(a) + \alpha 1_{\tilde{B}}$.
This is unital.