Talk 1 : John Dualé 27/1/2026

The Milnor–Švarc lemma, generalised:

The Milnor–Švarc lemma: $G \curvearrowright X$, where X is a proper geodesic space, and the action is proper discontinuous and cocompact, then

$$\mathrm{Cay}(G, S) \underset{QI}{\cong} X$$

finite generating set $\to S$; QI = quasi-isometry

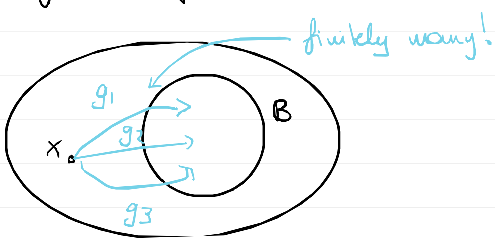
Aim: Weaken the assumptions and prove it still works!

Properly discontinuously $\to$ discontinuously

proper geodesic space $\to$ geodesic space

* But we have to assume G fin generated

Defn: Properly discontinuous: only finitely many $g \in G$ send x into the ball B

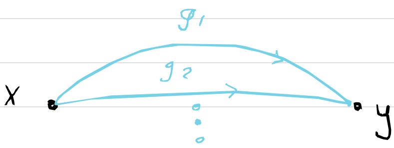


Defn: Discontinuous: We can now have infinitely many $g \in G$ sending x into B.

Discontinuity asserts that the points in the orbit of x don't "clump up".

Stabilisers: action prop disc $\to$ stabilisers are finite

action disc $\to$ stabilisers can be infinite



Claim: $g_2^{-1} g_1 x = x$

Proof: $d(g_2^{-1} g_1 x, x) = d(g_1 x, g_2 x) = y$ because we assumed $g_1, g_2 \in \mathrm{Stab}(x)$.

$\mathrm{Stab}(x) \supseteq \{g_i^{-1} g_j\}$

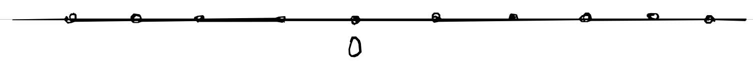
Conversely, if we have infinitely many $g \in G$ that send x to itself, we can make new (infinite) stabilisers by composing g with $g_i g_j$

Examples:

1) $G = \mathbb{Z}^2$, $X = \mathbb{R}$ $\quad G \curvearrowright X \quad (a,0)(x) = x+a, \quad (0,b)(x) = x$

Picture:

Stabilisers of 0 are all points $(0,b)$ ∞ many!



Claim: Action is cocompact + discontinuous but not properly discontinuous

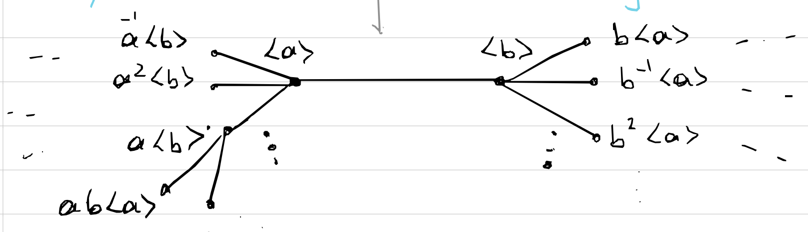
2) $G = F_2$, $X = T_{Bass-Serre}$ $\qquad$ note $F_2 \geq \langle a \rangle$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \geq \langle b \rangle$

Vertices are left cosets of the subgroups $\langle a \rangle$, $\langle b \rangle$

Edges: $\langle a \rangle$ •—• $a^i \langle b \rangle$

the word that comes before has to belong in $\langle a \rangle$

(edge since $e \in \langle a \rangle \cap \langle b \rangle$)



Action: $h\langle a \rangle \cdot g = gh\langle a \rangle$ $\quad$ Closed balls are not compact!

This is not a proper space or a proper disc. function! $\rightsquigarrow$ vertices have ∞ stabilisers $\rightarrow \mathrm{Stab}(\langle a \rangle) = \{g \in G \text{ st } g\langle a \rangle = \langle a \rangle\}$
$\iff g \in \langle a \rangle$
But $\langle a \rangle$ is infinite!

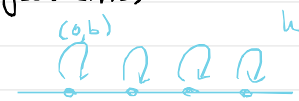
$\rightsquigarrow$ what about stabilisers of vertices of the form $h\langle a \rangle$?

$\mathrm{Stab}(h\langle a \rangle) = \{g \in G : gh\langle a \rangle = h\langle a \rangle\} \iff h^{-1}gh \in \langle a \rangle$

hence anything of the form, say $a\langle b \rangle a^{-1}$ stabilises $a\langle b \rangle$!

How can we find a way to relate the two geometries?

(i) Find maximum stabiliser subgroups $\quad$ for (i) $\rightsquigarrow$



hence it is $\langle (0,1) \rangle$

for (ii) $\rightsquigarrow$ $g\langle a \rangle g^{-1}$

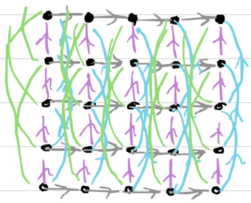
we get $g\langle a \rangle g^{-1}$ and $g\langle b \rangle g^{-1}$

(2) Choose one representative per conjugacy class.

(1) $\langle (0,1) \rangle$

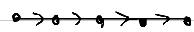
(2) $\langle a \rangle, \langle b \rangle$.

Now, for example 1, take the gen^ring^ set $\langle (1,0),(0,1) \rangle$, and take $\mathbb{Z}^2 = \langle (1,0),(0,1),(0,2),(0,3),\ldots \rangle$
and construct a Cayley graph for $\mathbb{Z}^2$ with respect to this new, infinite generating set!



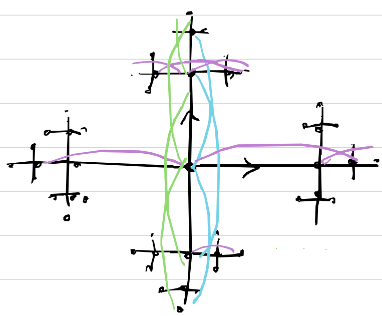
horizontally we have the same graph as before!

$\underset{QI}{\cong}$



idea: "coning off"

For example 2: $F_2 = \langle a,b \rangle$ but instead take $F_2 = \langle a, b, a^2, a^3, \ldots, b^2, b^3, \ldots \rangle$
and construct the Cayley graph wrt this generating set:



again, "coning off"

$\underset{QI}{\cong}$ Bass–Serre tree!

idea, everything in $\langle a \rangle$ is dist. 1 from each other
Same for everything in $\langle b \rangle$
Hence we send all points on horizontal lines to $\beta \langle a \rangle$ ($\beta \in \langle b \rangle$)
and vertical lines to $\alpha . \langle b \rangle$ ($\alpha \in \langle a \rangle$)

Charney - Crisp

Theorem: G fin. gen^ted^ ($= \langle T \rangle$), $G \curvearrowright X$, where X is a length space (just about weaker than geodesic space.)
and the action is discontinuous and cocompact.
Let S be the family of maximal stabiliser subgroups
Let $\mathcal{F}$ be one representative for each conj class in S.
Then $\mathrm{Cay}(G, T \cup \mathcal{F})$ is quasi-isometric to X.

Example: The trivial action $X \curvearrowright \{x\}$ $\quad \mathcal{F} = G$

$\{\text{vertices}\}$

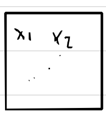
$Cay(G, G)$ is all the vertices with edges from anything $\in V$ to anything $\in V$.

Questions: Difference between coning off and quotienting?

If we quotient, we might not end up with a Cayley graph. With coning off, we don't act on the things we cone-off anymore, hence we still get a Cayley graph.

Some ideas from the proof of the theorem:

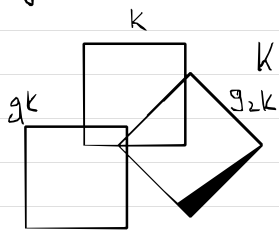
$\rightsquigarrow$ We want to show $|\mathcal{F}| < \infty$ "$\mathcal{F}$ is a finite set of subgroups".
ie finite number of conj. classes in our collection of stabilisers.



$K \subseteq X$

countably many x_i

$\{x_i\}$ converge to $x \in K$



K fundamental domain, we want the orbit of K to cover all stab.

This is true by cocompactness.

Suppose we have infinite $F_1, F_2, \dots$ st

$F_1 = \operatorname{stab}\{x_1\}$

$\vdots$

$F_i = \operatorname{stab}\{x_i\}$

$\vdots$

This will contradict discontinuity of our action:

$$* \quad d(g_i(x), x) \leq 2\underbrace{d(x_i, x)}_{\to 0}$$

hence $g_i \to x$

This contradicts discontinuity! Hence we cannot have infinitely many F_i.

This is true since $d(g_i(x), x) \leq \underbrace{d(g_i(x), x_i)}_{= d(g_i(x), g_i(x_i)) = d(x, x_i) \text{ by isometry}} + d(x_i, x)$

End of talk !!