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II.6. Functoriality

$\alpha : G \to G'$ a homom., F, F' proj. res.s of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ (resp.). Then F' is a cplx. of G-modules via α. F' acyclic (not proj. in general) over $\mathbb{Z}G$, so by fundamental lemma, we get an augmentation-preserving $\underbrace{G\text{-chain map}}_{\substack{\text{respects } G\text{-action:} \\ \tau(gx) = \alpha(g)\tau(x)}}$ $\tau : F \to F'$ (well-def. up to homotopy).

τ induces map $F_G \to F'_{G'}$ (well-def. up to homotopy), so get $\alpha_* : H_* G \to H_* G'$.

Prop.: Let $g_0 \in G$. $\alpha : G \to G$ be $\alpha(g) = g_0 g g_0^{-1}$. Then $\alpha_* : H_* G \to H_* G$ is id.

Pf
Take $\tau : F \to F$ to be $\tau(x) = g_0 x$.
Satisfies conditions + induces id. ▨

Cor.
G a gp., $N \trianglelefteq G$, then $G \curvearrowright N$ by conj. induces $G/N \curvearrowright H_* N$.

Ex. 1.

$N \trianglelefteq G$ as above. F a proj. res. of $\mathbb{Z}$ over $\mathbb{Z}G$, then F_N a cplx. of G/N-modules (Sean), so $H_* F_N$ inherits a G/N action. Show that $H_*(F_N) = H_* N$ and that the action on $H_* N$ is the same as the one above.

Soln.
F is also a ~~N-module~~ proj. res. over N so $H_* F_N = H_* N$.
[Hint: action of $g \in G$ on $H_* N$ comes from $\tau : F \to F$
$\tau(x) = gx$]
By another prev. exercise, G/N action on F_N is given by
$x \mapsto (gN)x = gx$

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