Deformation Theory Group

Talk 1

Seminar on Deformation Theory

Motivation. Consider a field $k = \bar{k}$, and a curve $C \subseteq \mathbb{P}^2_k$ of $\deg C = d$, i.e., $C = V(f)$ for some polynomial $f(x,y,z) = a_0 x^d + \ldots + a_n z^d$, $a_i \in k$, $n = \binom{d+2}{2} - 1$. If we consider $a = [a_0, \ldots, a_n] \in \mathbb{P}^n_k$, there is a variety $\mathcal{Y} \subseteq \mathbb{P}^2 \times \mathbb{P}^n$ given by $a_0 x^d + \ldots + a_n z^d = 0$ s.t. the projection $pr_1 : \mathcal{Y} \to \mathbb{P}^n$ gives a one-to-one correspondence between degree d curves and points in $\mathbb{P}^n$.

Usually, geometric objects come in families that we can parameterize by a scheme, i.e., there is a morphism $f : X \to T$ s.t. we want to understand his fibers (local study of parameter spaces). Our goal is to understand fibers of a morphism around a given base point.

Using deformation techniques we can study the following families of objects:

- Subschemes of a fixed scheme X.
- Coherent sheaves on a fixed scheme X.
- Deformations of abstract of schemes.

Question. What kind of morphisms $f : X \to T$ are a good notion of a family?

§1. Flat morphism.

History. Introduced by J. P. Serre in his paper GAGA to explain the relation between the local rings $\mathcal{O}_{X,x}$ (algebraic) and $\mathcal{H}_{X,x}$ (analytic).

Def. Let A be a ring, M an A-mod. M is flat over A if the functor

$$M \otimes_A - : A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}, \quad N \mapsto M \otimes_A N$$

is left exact, i.e.,

$$0 \to N_1 \to N_2 \to N_3 \to 0 \quad \text{exact}$$

$$\Rightarrow 0 \to M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3 \to 0 \quad \text{exact.}$$

If $f : A \to B$ is a ring morphism, we say B is flat over A if it is flat as an A-module.

Proposition.

(1) $M \in A\text{-}\underline{\text{mod}}$ is flat $\Leftrightarrow$ $\forall$ $I \subseteq A$ f.g.,

$I \otimes_A M \to M$ is injective.

(2) Base change: if $M \in A\text{-}\underline{\text{mod}}$ is flat, $f : A \to B$ ring morphism, $M \otimes_A B$ is a flat B-module.

(3) Transitivity: B flat A-alg, N flat B-mod, then N is a flat A-mod.

(4) Localization: $M \in A\text{-}\underline{\text{mod}}$ is flat $\Leftrightarrow$ $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$, $\forall$ $\mathfrak{p} \in \operatorname{Spec} A$.

(5) $0 \to M' \to M \to M'' \to 0$ exact, then

(a) M', M'' flat $\Rightarrow$ M flat

(b) M, M'' flat $\Rightarrow$ M' flat.

(6) If A is a noeth. local ring, M f.g. A-mod, then M flat $\Leftrightarrow$ M free.

Def. Let $f: X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-mod. We say $\mathcal{F}$ is flat over Y at $x \in X$ is $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-mod. (via the natural map $f^\#: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$)

$\mathcal{F}$ is flat over Y if it is flat $\forall x \in X$.

X is flat over Y if $\mathcal{O}_X$ is flat over Y.

Example. X noeth. scheme, $\mathcal{F} \in \underline{\mathrm{Coh}}(X)$, then $\mathcal{F}$ is flat over X $\Leftrightarrow$ $\mathcal{F}$ is loc. free

Prop. Let X, Y sch. of fin. type$/k$. $f: X \to Y$ flat. Let $y = f(x)$, $x \in X$. Then

$$\dim_x (X_y) = \dim_x X - \dim_y Y$$

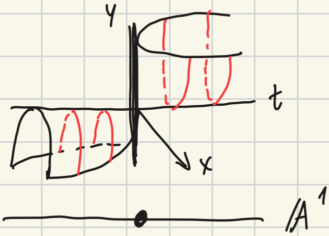
Example. $\varepsilon: \tilde{X} \to X$ blow-up of smooth subvar. on X smooth. Then ε is not flat.

Remark. The properties of irreducibility and reducedness are not preserved in flat families.

(*) Note: Over PID, flat $\iff$ torsion-free

For example, consider the flat family

$$\begin{array}{c} \operatorname{Spec} k[x,y,t]/\langle ty - x^2\rangle \\ f \downarrow \ (*) \\ \operatorname{Spec} k[t] \end{array}$$



We can see that $f^{-1}(0) \simeq \operatorname{Spec} k[x,y]/\langle x^2\rangle$, which is non-reduced. Also, in the family

$$\begin{array}{c} \operatorname{Spec} k[x,y,t]/\langle xy - t\rangle \\ f \downarrow \\ \operatorname{Spec} k[t] \end{array}$$

we have that

X_a is irred. for $a \neq 0$

$X_0 \simeq$  2 components

§2. Hilbert polynomial

Let k be a field, X proj. scheme $/k$, $\mathcal{O}_X(1)$ an ample line bundle. Then,

$$m \longmapsto \chi(X, \mathcal{O}_X(m))$$

is polynomial of degree $n = \dim X$, i.e., $\exists!\ P \in \mathbb{Q}[t]$ of $\deg P = \dim X$ s.t. $P(m) = \chi(X, \mathcal{O}_X(m))$ $\forall\ m \in \mathbb{Z}$.

Remark. $\mathcal{O}_X(1)$ ample $\Rightarrow$ $H^i(X, \mathcal{O}_X(m)) = 0$ $\forall\ m \gg 0$, $\forall\ i > 0$, so for $m \gg 0$, $P(m) = H^0(X, \mathcal{O}_X(m))$.

We call P the Hilbert polynomial of X w.r.t. $\mathcal{O}_X(1)$.

Example. (1) $C \subseteq \mathbb{P}^n_k$ smooth curve of genus g and degree d, by RR theorem

$$h^0(C, \mathcal{O}_C(m)) = h^0(\omega_C \otimes \mathcal{O}_C(-m)) + \deg(\mathcal{O}_C(m)) - g + 1$$

For $m > 2g-2$: $h^0(C, \mathcal{O}_C(m)) = dm - g + 1$

$\rightsquigarrow P(t) = dt - g + 1$.

(2) $H_d \subseteq \mathbb{P}^n$ hypersurface

$$0 \to \underbrace{\mathcal{O}_{\mathbb{P}^n}(-H_d)}_{\cong \mathcal{O}(-d)} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{H_d} \to 0$$

$$\rightsquigarrow P(t) = \binom{t+n}{n} - \binom{t-d+n}{n}.$$

Theorem. Let T be an integral noeth. sch. $X \subseteq \mathbb{P}^n_T$ closed subscheme, P_t Hilbert pol. of $X_t \subseteq \mathbb{P}^n_{k(t)}$. Then, X is flat over T iff P_t is independent of t.

Proof. We will prove the following:

(*) For any coherent sheaf $\mathcal{F}$ in $X = \mathbb{P}^n_T$, the Hilbert polynomial of $\mathcal{F}_t := \mathcal{F}|_{X_t}$ on $X_t = \mathbb{P}^n_{k(t)}$ is independent of t.

If we prove (*), we will obtain the result, since for $i : X \hookrightarrow \mathbb{P}^n_T$ closed embedding, we can take $\mathcal{F} = i_* \mathcal{O}_X$, notice that

$$H^0(X, \mathcal{O}_X(m)) = H^0(\mathbb{P}^n_k, \mathcal{F}(m))$$

and also $P_{X_t}(m) = h^0(X_t, \mathcal{O}_{X_t}(m))$ for $m \gg 0$.

Thus, we can assume $X = \mathbb{P}^n_T$ and consider $\mathcal{F} \in \underline{Coh}(\mathbb{P}^n_T)$. Moreover, question is local on T, so we consider $T = \operatorname{Spec}(A, \mathfrak{m})$ with A local noeth. ring.

We claim that the following assertions are equivalent:

(i) $\mathcal{F}$ flat over T.

(ii) $H^0(X, \mathcal{F}(m))$ free A-mod of fin. rank $\forall\, m \gg 0$.

(iii) P_t Hilbert pol. of X_t is indep. of t.

standard covering

(i) $\Rightarrow$ (ii) Čech coh.: $H^i(X, \mathcal{F}(m)) = h^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F}(m)))$.

$\mathcal{F}$ flat implies $C^i(\mathcal{U}, \mathcal{F}(m))$ is a flat A-mod$^{(*)}$, and $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$, $m \gg 0$ (Serre's vanishing, Hartshorne, Thm. III. 5. 2). This means there is an exact sequence

$$0 \to H^0(X, \mathcal{F}(m)) \to \underbrace{C^0(\mathcal{U}, \mathcal{F}(m))}_{\text{flat}} \to \dots \to \underbrace{C^n(\mathcal{U}, \mathcal{F}(m))}_{\text{flat}} \to 0$$

covering is finite

Breaking into several s.e.s. we see that $H^0(X, \mathcal{F}(m))$. Since $H^0(X, \mathcal{F}(m))$ is f.g. and flat A-mod, is free of finite rank.

direct sum is flat

(ii) $\Rightarrow$ (i) $S = A[x_0, \dots, x_n]$, $M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$ with $m_0 \gg 0$ s.t. $H^0(X, \mathcal{F}(m))$ are free, then $\mathcal{F} \simeq \widetilde{M}$ and $\mathcal{F}$ is flat over A.

(*) Use localization property.

(ii) $\Rightarrow$ (iii) We will prove that

$$P_t(m) = \operatorname{rk} H^0(X, \mathcal{F}(m)) \quad \text{for } m \gg 0.$$

Take $t \in T$ corresponding to $\mathfrak{p} \in \operatorname{Spec} A$, $T' = \operatorname{Spec} A_{\mathfrak{p}} \to T$. Consider the base change under $T' \to T$, we are reduced to the case in which $t \in T$ closed point. Denote the fiber X_t by X_0, $\mathcal{F}_t$ by $\mathcal{F}_0$ and we have $k(t) = k$ (flatness is preserved under base change). Take a presentation of k over A:

$$A^q \to A \to k \to 0 \qquad (*)$$

Taking the associated sheaf and tensoring by $\mathcal{F}$ we obtain

$$\mathcal{F}^q \to \mathcal{F} \to \mathcal{F}_0 \to 0$$

$\forall m \gg 0$

$$\rightsquigarrow \quad H^0(X, \mathcal{F}(m)^q) \to H^0(X, \mathcal{F}(m)) \to H^0(X_0, \mathcal{F}_0(m)) \to 0$$

(Hartshorne Ex. III.5.10)

On the other hand, tensoring $(*)$ by $H^0(X, \mathcal{F}(m))$, we obtain

$$H^0(X, \mathcal{F}(m))^q \to H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}(m)) \otimes_A k \to 0$$

$$\rightsquigarrow \quad H^0(X_0, \mathcal{F}_0(m)) \simeq H^0(X, \mathcal{F}(m)) \otimes_A k \quad \forall\, m \gg 0$$

$$\rightsquigarrow \quad \forall\, t \in T, \ H^0(X_t, \mathcal{F}_t(m)) \simeq H^0(X, \mathcal{F}(m)) \otimes_A k \quad \forall\, m \gg 0$$

(iii) $\Rightarrow$ (ii) Take $t \in T$ closed, $\eta \in T$ generic. Then, note that

$$H^0(X, \mathcal{F}(m)) \otimes_A K \simeq H^0(X_\eta, \mathcal{F}_\eta(m)) \simeq H^0(X_t, \mathcal{F}_t(m)) \simeq H^0(X, \mathcal{F}(m)) \otimes_A k$$

Thus, (ii) follows from the following lemma. ∎

Lemma (Hartshorne II. 8.9). $(A, \mathfrak{m})$ noeth. local domain, $A/\mathfrak{m} \simeq k$, $K = Fr(k)$. If M is f.g. A-mod, $\dim_k(M \otimes_A k) = \dim_K(M \otimes_A K) = r$ $\rightsquigarrow$ $M \simeq A^r$.

Corollary. $\dim X_t = \deg P_t$, $p_a(X_t) = (-1)^n (P(0) - 1)$ and $\deg(X_t) = n! \cdot (\text{lead. coeff. of } P_t)$ are constant in flat families.

Theorem. Let $f: X \to Y$ be a proper morphism of sch./k. Assume Y is regular, X is Cohen-Macaulay,

$$\dim X_y = \dim X - \dim Y \quad \forall y \Rightarrow f \text{ is flat.}$$

§ 3. Hilbert schemes

We would like to describe families of closed subschemes of a given scheme.

Theorem. Let k be a field, $Y \subseteq X = \mathbb{P}^n_k$ be a closed subscheme.

(a) $\exists$ H proj. scheme parametrizing closed subschemes of X with same Hilbert pol. P as Y, and $\exists$ $W \subseteq X \times H$ a universal subscheme, flat over H s.t. the fibers of W over closed points $h \in H$ are all closed subschemes of X with Hilbert pol. P.
Furthermore, H is universal: if T is any scheme, $W' \subseteq X \times T$ is a closed subscheme flat over T, whose fibers are closed subschemes of Y with Hilbert pol. P, then $\exists !$ $T \to H$ s.t. $W' = W \times_H T$.

(b) If $y \in H$ corresponds to $Y \subseteq X$,
$T_y H \cong H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X} \simeq \mathcal{H}om(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$
is the normal sheaf of Y in X.

(c) If Y is a l.c.i and $H^1(Y, \mathcal{N}_{Y/X}) = 0$,
H is smooth at y of dim. $h^0(Y, \mathcal{N}_{Y/X})$.

(d) If Y is l.c.i, $\dim_y H \geq h^0(Y, \mathcal{N}_{Y/X}) - h^1(Y, \mathcal{N}_{Y/X})$.

Idea of (a).

$\{$subschemes of $\mathbb{P}^n_k\} \longleftrightarrow \{$subspaces of $k[x_0, \ldots, x_n]\}$

(i) Replace RHS by finite-dim. Grassmannian.

(ii) Image is a subvariety.

Let's talk about (i). Take $\mathcal{I}_Y$ the ideal sheaf of $Y \subseteq \mathbb{P}^n_k$, i.e.,

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$$

Thm. $\forall P \in \mathbb{Q}[t]$, $\exists N(P) \in \mathbb{N}$ s.t. if $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_k}$ has Hilbert pol. P, then $\forall n \geq N(P)$

(a) $h^i(\mathbb{P}^n, \mathcal{I}(n)) = 0 \quad \forall i \geq 1$
(b) $\mathcal{I}(n)$ globally generated
(c) $H^0(\mathbb{P}^n, \mathcal{I}(n)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \twoheadrightarrow H^0(\mathbb{P}^n, \mathcal{I}(n+1))$

Take $N = N(P)$, then $H^0(\mathbb{P}^n, \mathcal{I}_Y(N)) \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N))$ determines Y, since (because of (b))

$$\mathcal{I}_Y(N) = \operatorname{im}\left(\mathcal{O}_{\mathbb{P}^n} \otimes H^0(\mathbb{P}^n, \mathcal{I}_Y(N)) \hookrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N)) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}(N)\right)$$

Thus, N work $\forall Y \subseteq \mathbb{P}^n_k$ with Hilbert pol. P, so for $Q(N) = h^0(\mathbb{P}^n_k, \mathcal{I}_Y(N))$ there is an injective map

$$\left\{ \begin{array}{l} \text{subschemes of } \mathbb{P}^n \text{ with} \\ \text{Hilbert pol. } P \end{array} \right\} \hookrightarrow Gr(Q(N), h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N)))$$

Examples. (1) $\mathrm{Hilb}^1(X) \simeq X$.

(2) $\mathrm{Hilb}^P(X \times_S Z / Z) \simeq \mathrm{Hilb}^P(X/S) \times_S Z$