Exercises (chapter. subchapter. exercise #).

II.2.1) If S is an arbitrary G set, show $(\mathbb{Z}S)_G \cong \mathbb{Z}[S/G]$.

- We know $M_G \cong M \otimes_{\mathbb{Z}G} \mathbb{Z}$ for any G-module M. Therefore $-_G$ splits over $\oplus$ i.e. $(M \oplus M')_G \cong (M \oplus M') \otimes_{\mathbb{Z}G} \mathbb{Z} \cong M_G \oplus M'_G$.

We have that $\mathbb{Z}S$ splits as $\mathbb{Z}S \cong \bigoplus_{\text{orbits of } G \curvearrowright S} \mathbb{Z}S_i$,

So $(\mathbb{Z}S)_G \cong \bigoplus_{\text{orbits}} (\mathbb{Z}S_i)_G \cong \bigoplus_{\text{orbits}} \mathbb{Z}$ since every basis element in $\mathbb{Z}S_i$ is identified under $G \curvearrowright \mathbb{Z}S_i$.

Proposition 2.4) Let X be a free G-complex and let $Y := X/G$. Then $C_*(X)_G \cong C_*(Y)$

II.2.2.) Show we can relax "free" in the above.
The G-action on a free G-complex freely permutes the cells. So, the G action on $S := \{n\text{-cells of } X\}$ is free. We then get an isomorphism of $(\mathbb{Z}S)_G$ with $\mathbb{Z}\{n\text{-cells of } Y\} \cong \bigoplus_{\substack{\text{orbits} \\ \text{of } n\text{-cells} \\ \text{in } X.}} \mathbb{Z}$ for each n.

The above shows the action need not be free, just a discrete action on a set.
Note that in the proof of Prop. 2.4. we use a trick to make $G \curvearrowright \{n\text{-cells}\}$ a permutation. $C_n(X)$ is generated by oriented n-cells in X, but since an orbit of n-cells is just a copy of G (by free action), we can choose an orientation on this orbit of cells so that G acts by $\sigma \longmapsto \sigma'$ for generators (n-cells) $\sigma, \sigma' \in C_n(X)$. In principle, it is possible that some $g \in G$ acts as $\sigma \longmapsto -\sigma'$, if we were to choose the orientation of σ' in such a way. In this case, there is more data than just a permutation on the n-cells, so we can't use II.2.1. directly.

Suppose that G acting on an orbit of n-cells has a non-trivial stabiliser. i.e. $\exists\ g \in \mathrm{Stab}(\sigma) \setminus \{1\}$. If g acted on σ such that on the boundary it was a degree -1 map $S^{n-1} \to S^{n-1}$, then g would be a non-trivial automorphism on the factor of $\mathbb{Z}$ corr to σ in $C_n(X)$ (the automorphism $1 \mapsto -1$).

Otherwise, we have an action who's data on cells is just a permutation, so we use II.2.1.

A: If G is an action which does not have cell inversions. i.e. if $(\sigma \overset{g}{\mapsto} \sigma \Rightarrow \deg(g : S^{n-1} \to S^{n-1}) = 1)$, then prop. 2.4 holds.

II.2.3. If $H \trianglelefteq G$ and M a G-module.

a) Show $\exists$ action $G/H \curvearrowright M_H$. ie M_H is a G/H module.

Define $G/H \ni gH \cdot m + I_H = gm + I_H$ where I_H is the ideal generated by $\{hm - m \mid \forall h \in H, m \in H.\}$.

We have $ghH = Hhg = hgH$ because H is normal.

So $ghH \cdot (m + I_H) = hgH \cdot (m + I_H) = hgm - \underbrace{(hgm - gm)}_{\in I_H} + I_H$

$= gH \cdot (m + I_H)$

So, the action is well defined. It is an action by $G \curvearrowright M$ being an action.

b) Show $M_G \cong (M_H)_{G/H}$.

Elements of $(M_H)_{G/H}$ are $(m + I_H) + I_{G/H}$ where $I_{G/H}$ is the ideal generated by $\{gH \cdot (m + I_H) - (m + I_H)\}$ wher $gH \in G/H$ and $\cdot$ is as above. We have

$gH \cdot (m + I_H) - (m + I_H) = (gm - m) + I_H$. So $I_{G/H}$ is generated by $m' + I_H$ where $m' \in \ker(M \to M_G)$.

Therefore the map $m + I_G \longmapsto (m + I_H) + I_{G/H}$ is

1) well defined ($gm - m + I_G \longmapsto 0 \Rightarrow gm + I_G \longmapsto x \longleftarrow gm + I_G$).

2) An injection

It is clearly surjective.

§ II.3.1

Let $\{g_1, \dots, g_n\} \subseteq G$ all pairwise commute.

Let $z := \sum_{\sigma \in S_n} (-1)^{\operatorname{Sign}(\sigma)} [g_{\sigma(1)} | \dots | g_{\sigma(n)}] \in C_n(G)$.

S_n "Sym gp on n letters".

Show $\partial(z) = 0$ (z a cycle).

$$\partial = \sum_{i=0}^{n} (-1)^i d_i \quad \text{where} \quad d_i[g_1|\dots|g_n] = \begin{cases} [g_2|\dots|g_n] & i = 0 \\ [g_1|\dots|g_i g_{i+1}|\dots|g_n] & 0<i<n \\ [g_1|\dots|g_{n-1}] & i = n. \end{cases}$$

Let $* \notin \{g_1, \dots, g_n\}$. Define $[*|x_1|\dots|x_n] = [x_1|\dots|x_n]$

$[x_1|\dots|x_i|*|x_{i+1}|\dots|x_n] = [x_1|\dots|x_i x_{i+1}|\dots|x_n]$ and

$[x_1|\dots|x_n|*] = [x_1|\dots|x_{n-1}]$.

Let $\{\hat{g}_0, \hat{g}_1, \dots, \hat{g}_n\}$ be such that $\hat{g}_0 = *$ & $\hat{g}_i = g_i$ for $1 \le i \le n$.

Then we observe $\partial[g_1|\dots|g_n] = \sum_{\tau \in J} \operatorname{Sign}(\tau) [\hat{g}_{\tau(0)}, \hat{g}_{\tau(1)}, \dots, \hat{g}_{\tau(n)}]$

where $J := \{(0,1), (0,2), \dots, (0,n)\} \subseteq S_{n+1}$ "Sym. gp. on $n+1$ letters".

So $\partial z = \sum_{\tau \in J} \sum_{\sigma \in S_n} \operatorname{Sign}(\tau) \operatorname{Sign}(\sigma) [\hat{g}_{\tau\sigma(0)}, \dots, \hat{g}_{\tau\sigma(n)}]$

$$= \sum_{\tau \in J} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{Sign}(\tau\sigma) [\hat{g}_{\tau\sigma(0)}, \dots, \hat{g}_{\tau\sigma(n)}]$$

$$= \sum_{\eta \in \mathfrak{S}_{n+1}} \operatorname{Sign}(\eta) [\hat{g}_{\eta(0)}, \dots, \hat{g}_{\eta(n)}]$$

Now suppose $\eta \in \mathfrak{S}_{n+1}$ is s.t. $\eta(i) = 0$ with $0 < i < n$. Then $\eta' := (i-1, i+1)\eta$ is s.t.

$$[\hat{g}_{\eta(0)}, \dots, \hat{g}_{\eta(i-1)}, \hat{g}_{\eta(i)}, \hat{g}_{\eta(i+1)}, \dots, \hat{g}_{\eta(n)}] =$$

$$[\hat{g}_{\eta(0)}, \dots, \hat{g}_{\eta(i-1)}, -, \hat{g}_{\eta(i+1)}, \dots, \hat{g}_{\eta(n)}] =$$

$$[\hat{g}_{\eta(0)}, \dots, \hat{g}_{\eta(i-1)}\hat{g}_{\eta(i+1)}, \dots \hat{g}_{\eta(n)}] =$$ (since the g_i pairwise commute).

$$[\hat{g}_{\eta(0)}, \dots, \hat{g}_{\eta(i+1)}\hat{g}_{\eta(i-1)}, \dots \hat{g}_{\eta(n)}] =$$

$$[\hat{g}_{\eta'(0)}, \dots, \hat{g}_{\eta'(i-1)}\hat{g}_{\eta'(i+1)}, \dots \hat{g}_{\eta(n)}] =$$

$$[\hat{g}_{\eta'(0)}, \dots, \hat{g}_{\eta'(i-1)}, \hat{g}_{\eta'(i)}, \hat{g}_{\eta'(i+1)}, \dots \hat{g}_{\eta'(n)}].$$

Furthermore, $\operatorname{Sign}(\eta) = -\operatorname{Sign}(\eta')$, so these terms cancel in the sum.

We find pairs of cancelling terms also when $\eta(0) = 0$ or $\eta(n) = 0$.

- II.6.1 Let Y be a path connected space. If Y has contractible universal cover X, with deck group G, show $H_*(Y) \cong H_*(G)$.

X is the universal cover for Y and $G \curvearrowright X$ freely. Consider the singular chain cx. $C^{\text{sing}}(X)$, where $C_n^{\text{sing}}(X)$ is generated by $\sigma^n : \Delta^n \longrightarrow X$.
There is a G-action $\sigma^n \overset{g}{\longmapsto} g \circ \sigma^n$.

The chain maps of $C^{\text{sing}}(X)$ involve these ∂_i maps

$$\partial_i : (\Delta^n \to X) \longrightarrow (\Delta^{n-1} \to X) \qquad [v_1, \ldots, v_{n-1}] \longmapsto [v_1, \ldots, \underset{\substack{\uparrow \\ \text{position } i}}{0}, \ldots, v_{n-1}]$$
$$\sigma \longmapsto \sigma \circ (\Delta^{n-1} \hookrightarrow \Delta^n)$$

and the G-action is compatible with these ∂_i, i.e, $g \cdot \partial_i \sigma = \partial_i \, g \cdot \sigma$.
So, the $C_n^{\text{sing}}(X)$ are free $\mathbb{Z}G$ modules and since $X \simeq *$, $C^{\text{sing}}(X)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Then the usual argument...
As a $\mathbb{Z}G$-module, $C_n^{\text{sing}}(X)$ is generated by G orbits of n-simplices.

$$C_n^{\text{sing}}(X) \cong \bigoplus_{\text{orbits}} \mathbb{Z}G. \quad \text{So } \left(C_n^{\text{sing}}(X)\right)_G \cong \bigoplus_{\text{orbits}} \mathbb{Z}.$$

The covering map $p: X \longrightarrow Y$ induces a map

$$p_* : C^{\text{sing}}(X) \longrightarrow C^{\text{sing}}(Y) \quad \sigma^n \longmapsto p \circ \sigma^n.$$
and $p_*(g \cdot \sigma^n) = p \circ g \circ \sigma^n = p \circ \sigma^n = p_*(\sigma^n)$

So p_* is a G-module morphism if we give $C^{\text{sing}}(Y)$ the trivial action.

$$(p_*)_G : (C^{sing}(X))_G \longrightarrow (C^{sing}(Y))_G = C^{sing}(Y)$$

is an iso if we compare bases. So

$$H_*(G) = H_*(C^{sing}(X)_G) = H_*(C^{sing}(Y)) = H_*(Y).$$

7) Amalgamated Products.

Amalgamation diagram $\in$ Grp.

$$\begin{tikzcd} A \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & G_2 \arrow[d, "\beta_2"] \arrow[ddr, bend left] & \\ G_1 \arrow[r, "\beta_1"] \arrow[drr, bend right] & G \arrow[dr, dotted, "\exists !"] \arrow[ul, phantom, "\ulcorner"] & \\ & & H \end{tikzcd}$$

G is the amalgamated product $G \cong G_1 *_A G_2$ iff this diagram is a pushout

$$G \cong \frac{G_1 * G_2}{\alpha_1(a) = \alpha_2(a) \; \forall a \in A.}$$

Usually α_1, α_2 injective. So A is a subgroup of G_1 & G_2.

Van-Kampen (for CW-complexes).

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, hook] & X_2 \arrow[d, hook] \\ X_1 \arrow[r, hook] & X \end{tikzcd}$$

All maps cellular, $X = X_1 \cup_Y X_2$

Then $\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$

Want to realise any (injective) amalgamation diagram via $K(\pi,1)$s ie. make all the above spaces $K(\pi,1)$s, with correct fundamental groups.

Lemma (1): If α_1, α_2 injective $\Rightarrow \beta_1, \beta_2$ injective.

Lemma (2): Let $i : X' \hookrightarrow X$ be an inclusion of CW-complexes s.t. $i_* : \pi_1(X') \to \pi_1(X)$ is injective.

Let $p : \tilde{X} \to X$ be a universal cover of X, the each connected component of $p^{-1}(X')$ is simply connected, i.e. is a universal cover for X'.

Proof. Let $\tilde{X}_i'$ be a connected component of $p^{-1}(X')$. Then we have

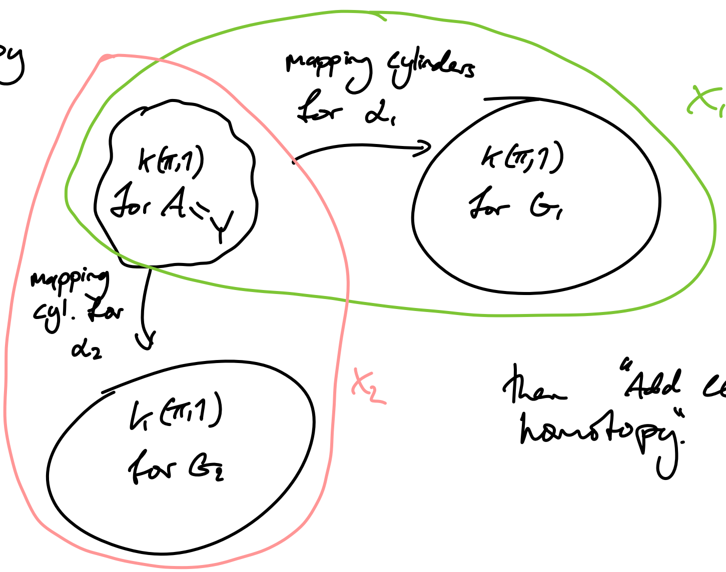
$$\begin{tikzcd} \pi_1(\tilde{X}_i') \arrow[r, "i_*"] \arrow[d, "p_*"'] & \pi_1(\tilde{X}) = 1 \arrow[d, "p_*"] \\ \pi_1(X') \arrow[r, hook, "i_*"] & \pi_1(X) \end{tikzcd}$$

homotopy lifting prop. $\Rightarrow p_*$ is injective

So i_* up top is also injective, so $\pi_1(X') = 1$.

lemma: We can realise $G_1 \xleftarrow{\alpha_1} A \xrightarrow{\alpha_2} G_2$ in $K(\pi,1)$s. i.e. $\exists$ $K(\pi,1)$s. $X_1 \hookleftarrow Y \hookrightarrow X_2$

By



then "Add cells killing higher homotopy"

Now we construct $X = X_1 \cup_A X_2$.

By Van-Kampen $\pi_1(X) = \pi_1(X_1) \cup_{\pi_1(Y)} \pi_1(X_2)$ (as required)

Let $p : \tilde{X} \to X$ be uni. cover.

$\pi_1(Y) \hookrightarrow \pi_1(X)$, so by lemma (2) $p^{-1}(Y)$ has conn. comp. which are uni covers for Y.

Also, by lem. (1) $\pi_1(X_i) \hookrightarrow \pi_1(X)$, so similarly $p^{-1}(X_i)$ has conn. comp. uni covers for X_i

Choose conn. componet so we have

$$\begin{tikzcd} \tilde{Y} \arrow[r, hook] \arrow[d] & \tilde{X}_2 \arrow[d] \\ \tilde{X}_1 \arrow[r, hook] & X \end{tikzcd}$$

$\leftarrow$ uni cover

uni covers

Since Y, X_1, X_2 are $K(\pi,1)$s we have

$$H_i(\tilde{Y}) = H_i(\tilde{X}_i) = 0 \text{ for } i > 0$$

MV Seq.

$$\cdots \to \underset{0}{\underbrace{H_2(\tilde{Y})}} \to \underset{0}{\underbrace{H_2(\tilde{X}_2) \oplus H_2(\tilde{X}_1)}} \to H_2(\tilde{X}) \to H_1(\tilde{Y}) \to \cdots$$

$H_1(\tilde{Y}) = 0$

$\Rightarrow H_i(\tilde{X}) = 0$ for all $i > 0$

So X is a $K(\pi,1)$ for $G_1 *_A G_2 = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$

So applying MV to these $K(\pi,1)$s, we get a a MV sequence in group homology.

$$\cdots \to H_n(A) \to H_n(G_1) \oplus H_n(G_2) \to H_n(G) \to H_{n-1}(A) \to \cdots$$

Homology & Cohomology w/ coefficients

$M \otimes_R N$ is defined whenever $M \in \mathrm{Mod}_R$ & $N \in {}_R\mathrm{Mod}$

Want $m \otimes (rs)n = m(rs) \otimes n = (mr)s \otimes n = mr \otimes sn$
$= m \otimes r(sn) = m \otimes (rs)n$.

if $M \in {}_R\mathrm{Mod} \Rightarrow$
$m \otimes (rs)n = (rs)m \otimes n = r(sm) \otimes n = sm \otimes rn = m \otimes (sr)n$ #.
$(= M \otimes N)$.

Recall $M \otimes_R N$ is $M \otimes_{\mathbb{Z}} N \big/ {mr \otimes n = m \otimes rn}$

For group actions, we can avoid having to consider L/R modules since any L $G \curvearrowright M$ is also a R $M \curvearrowleft G$ by precomposing with the anti-automorphism $g \mapsto g^{-1}$

So if M, N are two left G-Modules, we can make sense of $M \otimes_{\mathbb{Z}G} N$ (denoted $M \otimes_G N$).

$M \otimes_G N$ is $M \otimes_{\mathbb{Z}} N \big/ {g^{-1}m \otimes n = m \otimes gn} = M \otimes N \big/ {m \otimes n = gm \otimes gn}$ (beware typo!).

So $M \otimes_G N = (M \otimes N)_G$ where $G \curvearrowright M \otimes N$ diagonally.

So... $M \otimes_G N \approx N \otimes_G M$

We also define an action of G on $\mathrm{Hom}(M,N)$ where $(gu)(m) := g \cdot u(g^{-1}m)$.

ASK GROUP ABOUT PRECEDING PARAGRAPH.

If $gu = u \iff g \cdot u(g^{-1}m) = u(m) \quad \forall m \in M \ \forall g$

$\iff u(g^{-1}m) = g^{-1}u(m) \iff u \in \mathrm{Hom}_G(M,N)$.

So $\mathrm{Hom}_G(M,N) = \mathrm{Hom}(M,N)^G \leftarrow$ denotes fixed points.

- Defining $H_*(G,M)$ & $H^*(G,M)$.

Let F be a proj resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. M a G-module. Define homology w/ coefficients in M

$$H_*(G,M) := H_*(F \otimes_G M)$$

Where $F \otimes_G M$ looks like

$$\cdots \; F_n \otimes_G M \xrightarrow{f_n \otimes id_M} F_{n-1} \otimes_G M \xrightarrow{f_{n-1} \otimes id_M} \cdots$$

(Not necessarily exact!)

$F \otimes_G M$ can also be thought of as a tensor product of chain complexes where

$$M = \cdots \to 0 \to 0 \to \underset{M_0}{M} \to 0$$

But, this is odd, seeing as F is a <u>projective</u> resolution of $\mathbb{Z}$, and M (as above) is just a chain cx. (over $\mathbb{Z}G$)
What if we demanded also a projective resolution of M, i.e. some $\eta: P \to M$ and $\varepsilon: F \to \mathbb{Z}$ and set

$$H_*(G,M) = H_*(F \otimes_G P)$$

This is an equivalent definition to

$$H_*(G,M) := H_*(F \otimes_G M)$$

because $Id_F \otimes \eta : F \otimes_G P \to F \otimes_G M$ is a weak equiv.

(Book uses $F \otimes \eta$ instead of $Id_F \otimes \eta$)

Also, $\varepsilon \otimes Id_P : F \otimes_G P \to \mathbb{Z} \otimes_G P$ is a weak equiv.

Thus $H_*(G,M) = H_*(P_G)$.

We can now compute $H_0(G,M)$

Using $F_1 \to F_0 \to \mathbb{Z} \to 0$ & R-exactness of $\_\otimes_G M$

$$\Rightarrow \quad F_1 \otimes_G M \to F_0 \otimes_G M \to \underset{\substack{\shortparallel \\ M_G}}{\mathbb{Z} \otimes_G M} \to 0 \quad \text{exact.}$$

$$\Rightarrow \quad H_0(G,M) = M_G$$

NB. H_* denotes homology of

$$F_1 \otimes_G M \to F_0 \otimes_G M \to 0 \to 0$$

(i.e. <u>not</u> reduced homology).