SW inequalities of symplectic manifolds

Overview: ① Recap the Weitzenböck
② Existence of symplectic structures
③ Non-vanishing
④ J-holomorphic curves / Gromov invariants
⑤ Idea of proofs

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① Review

$(X, \underset{\sim}{\omega}, \mathcal{J})$

→ real vector SV bdl. $\mathrm{Sym}^2(\Lambda^1_{\mathbb{R}}) \to \mathrm{Sym}^2(\Lambda^{1,0}_{\mathbb{C}})$
→ Hermitian-Einstein interpretation
• $g \in \mathrm{Sym}^2$: positive real scalar products $\leftrightarrow$ (type-1 antisymmetry)
$\leftrightarrow$ $\exists$ $\mathbb{C}^n \cong \mathbb{R}^{2n}$ s.t. g acts by $\mathbb{C}$-linear maps
$\leftrightarrow$ $\exists$ $\mathbb{C}$-basis of $\Lambda^{1,0}$ s.t. g is diagonal
• $\omega \in \Lambda^2$: symplectic (skew-symmetric) $\leftrightarrow$ pairing SW isomorphic
• $\omega \in \Lambda^{1,1}$: symplectic and type-1 $\leftrightarrow$ pairing SW isomorphic
• $\omega \in \Lambda^{1,1}$: symplectic and type-1 $\leftrightarrow$ $\exists$ $\mathbb{C}$-basis of $\Lambda^{1,0}$ on which ω is $\mathbb{C}$-linear

Rmk: An Hermitian manifold with $h^{2,0} \geq 1$ cannot be Kähler type.

Natural question: When does (X, ω) admit a compatible complex structure?

Def: X symplectic manifold. A complex structure J on X is said to be compatible with ω if
• J is an $\mathbb{R}$-linear map $TX \to TX$ s.t. $J^2 = -\mathrm{id}$
• $\omega(Ju, Jv) = \omega(u, v)$ for all $u, v \in TX$
• $g_J(u, v) := \omega(u, Jv)$ is a Riemannian metric.

Rmk: (X, ω) admits two compatible ACS, J & $\overline{J}$ with $\overline{J} = -J$.

$\mathcal{J}(X, \omega)$ is contractible

In fact we cannot guarantee. Rmk $\mathcal{J}$ is integrable!

Consequence: (X, ω) has a well-defined $c_i(X, \omega) = c_i(TX, J)$ for $J \in \mathcal{J}(X, \omega)$

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① Existence of symp. structures

Question: When does X have a symplectic form?

↳ obvious necessary conditions:
• $\exists\ \theta \in H^2(X, \mathbb{R})$ s.t. $\theta \cup \theta \neq 0$
• $\exists$ an almost complex structure.

Stalkers: When do ACS exist?

Theorem: [Wu] X stable, 6-mfd, $c_1 = 0$ $\Rightarrow$ $c_3(X) = 0$

X has an ACS with $c_1 = 0$ $\Leftrightarrow$ $\begin{cases} c_1 = 0 \in H^2(X, \mathbb{Z}) \\ \alpha \cup \alpha \neq 0 \ \forall\ \alpha \neq 0 \in H^2(X, \mathbb{R}) \end{cases}$

Rmk: characteristic classes

Example: $X = \mathbb{P}^3$ or $\overline{\mathbb{CP}^3}$ has an ACS since $X \cong \overline{X}$ via $\mathbb{CP}^3 \cong \mathbb{CP}^3$ and not symplectic

↳ Hirzebruch signature theorem

Rmk: An almost complex 6-manifold always has b^3 odd.

Ex: c_1 is a characteristic element in the following form $\Leftrightarrow$ $c \cdot c - \sigma = 0 \bmod 8$

$$\underbrace{E_8 \oplus \cdots \oplus E_8}_{k} \oplus \underbrace{H \oplus \cdots \oplus H}_{\ell}$$

Question: X 4-manifold

Suppose X admits almost complex structure $J \in \mathcal{J}(X, \omega)$. Then
• $\omega \wedge \omega \neq 0$, so ω is a volume form
• $c_1(X, J) = 0$
• $\exists$ a symplectic form on X i.e. $\exists\ \omega \in \Omega^2(X)$ s.t. $d\omega = 0$ and $\omega \wedge \omega > 0$

Does X have a symplectic form or not? $\{\omega\} = 0$ or $c_1(X, \omega) = c$?

Recipe: Ehr. $\mathbb{C}$-3: hypersurface $X_d \subset \mathbb{CP}^3$ ($d \geq 5$), must have $\int_X c_1(TX) < 0$.
(Consistent exhaustively + adjunction inequality)

Naive question: X, d, c as above

Could it happen that X has no symplectic forms at all?

→ Uhlenbeck before SW.

Recipe: Y_{2d}, e.g. $\mathbb{CP}^3$ if $\mathbb{CP}^3$ (Taubes)

→ SW monotonicity vanishes (PDE)
→ Taubes' non-vanishing theorem for symplectic mfd. Y

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② Non-vanishing

(X, ω) symplectic, for each $J \in \mathcal{J}(X, \omega)$ we can define $\underset{\sim}{\omega}_J$

↳ independent of J up to isomorphism (contractible)

In general spin^c structure ξ twisted with $E \to X$

$$L_\xi = K^{\frac{1}{2}} \otimes E^{\mathbb{C}^n} \qquad K = \Lambda^{n,0} \ \text{spin}^c \ \text{canonical bundle (canonical to } J\text{)}$$

Additive relation $2E \to K$

→ complex conjugation: $\overline{L_\xi} \cong L_{\overline{\xi}}$ $\qquad \overline{L_\xi} \cong L_\xi$

Theorem: [Lee] $\{\omega\}$ symplectic $\Leftrightarrow$ $\begin{cases} \mathrm{SW}(X, \overline{\omega}_J) = \pm 1 \\ \text{if } b^1 > 1 \ \mathrm{SW} = 0 \end{cases}$

Moreover: • $b^+ \geq 2$: $\mathrm{SW}(X, \xi) \neq 0 \Rightarrow \exists\ \omega \leq (\xi)$ (ξ-tame) $\geq c_1(\xi) \cdot [\omega]$ equality iff $\xi \in \mathcal{S}^{can}$
• $b^+ = 1$: $\mathrm{SW}^{\mathcal{P}}(X, \xi) \neq 0 \Rightarrow \begin{cases} \exists\ c_1(\xi) \cdot [\omega] \\ \mathrm{SW}^{\mathcal{P}}(X, \xi) \neq 0 \Rightarrow c_1(\xi) \cdot [\omega] < 0 \end{cases}$

Remark: uncalculable?

Immediate consequences:
• Symplectic 4-mfds with $b^+ \geq 2$ do not admit metrics of PSC
$\Rightarrow$ $\#\ \mathbb{CP}^2$ $\#$ $\overline{\mathbb{CP}^2}$ is not symplectic unless $b = 1$
• If $X \neq S, \# T_X$ with $b^1(X) \geq 1$ and $b^+(X), \pm 1$, then X is not symplectic

Generalized by Kotschick: $X = V_1 \# V_2$ symplectic only manifold,
$\Rightarrow$ V_1 or V_2 is a homology sphere. (X is irreducible)

Corollary: If X has a symplectic form with $c_1(X) = 0$, then X is a flat 4d and symplectic torus or $\mathbb{T}^4$.

↳ e.g. this holds for the $\mathbb{T}^4$ and the K3 surface

Another application to the Thom conjecture.

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③ Relation to J-holomorphic curves

Def: (X, ω) symplectic, $J \in \mathcal{J}(X, \omega)$

A J-holomorphic curve in X is $f : (\Sigma, j) \to (X, J)$ from a Riemann surf. (Σ, j) s.t. $df \circ j = J \circ df$.
$\Sigma \subset \iota(\Sigma)$
↳ they coincide in symplectic topology! (Gromov 1985)

Rmk: These curves are not necessarily embedded:
→ multiply covered
→ self-intersections
→ singularities
} bad behavior on $\mathbb{CP}^2$ via embedded curves

• Special property in 4d:

→ positivity of intersections: $C \cdot C > 0$ for algebraic connected simple J-hol curves

→ adjunction formula for embedded curves: C embedded, $w = \mathbb{C}(C) \in H_2$

$$2g - 2 = C \cdot C + c_1(K) \cdot C$$

→ for general J, the moduli space of connected embedded J-curves in w class $H^{\mathrm{reg}}(X, J, w) \cong w \cdot C - c_1(K) \cdot w$.

Theorem: (Taubes)

(X, ω) symplectic $\qquad \mathrm{SW}(X, \xi) \neq 0 \qquad (b^+ > 1, \ \mathrm{SW}^{\mathcal{P}})$

$\Rightarrow$ $c_1(\xi)$ is Poincaré dual to an embedded symplectic surface $C \subset X$
$[C] = PD(c_1(\xi))$ (and consistently oriented)

Moreover, each component C_i satisfy $c_1(\xi) \cdot [C_i] \geq g(C_i) - 1 \leq C_i \cdot C_i$.

Corollary: If $b^+ > 1$, the canonical class $c_1(K)$ is Poincaré dual to a symplectic surface.
↳ A symplectic version of the "canonical divisor" in AG

Corollary: (Y, ω) minimal $\qquad \mathrm{SW}(Y, \xi) \neq 0$, $\mathrm{SW}(Y, \xi) \neq 0$ (then) and $c_1(\xi) \cdot c_1(\xi) > 0$ $\Rightarrow$ $\exists$ $\mathrm{SW}^{\mathcal{P}} \neq 0$, $c_1(\xi) \cdot c_1(\xi) > 0$

Proof: By positivity of intersection: $C_+ \cdot C_+ = -C_+ \cdot C_-$
and self-intersection of spheres ≤ -1

Corollary: (Y, ω) symplectic with $b^+ \geq 2$, then it has simple type.

Proof: $\mathrm{SW}(Y, \xi) \neq 0 \Rightarrow w_0 \mathrm{SW}(X, \xi_{c_1}) \neq 0$ by duality

$$\Rightarrow c_1(\xi) \cdot c_1(K - \xi) \leq 0$$

Moreover, $\dim \mathcal{M}(X, \xi) = \frac{c_1(\xi) \cdot c_1(\xi) - (2\chi + 3\sigma)}{4}$ $\qquad \frac{c_1(\xi) \cdot c_1(\xi) - c_1(K)^2}{4}$

$$= \cdots = c_1(\xi) \cdot c_1(\xi - K) \geq 0$$

$\Rightarrow$ $\dim \mathcal{M} = 0$

Some consequences:

(a) Symplectically minimal CP^2 smoothly minimal

Corollary: (Taubes) (Y, ω) symplectic, $b^+ \geq 2$

Suppose Y contains a smoothly embedded sphere S with $S \cdot S = -1$.

Then X contains a J-hol. embedded sphere C with $C \cdot C = -1$ and $[C] = [S]$ or $[C] = -[S]$. Moreover in that case $c_1(K) \cdot [S] = \pm 1$.

→ Don't work lately with $b^+ = 1$

Rmk: use Taubes's theorem to find a symplectic surface in $[S]$
↳ blow that it is a sphere

Blow-up formulas for Seiberg-Witten invariants

(2) Uniqueness of symp structures on CP^2

Corollary: (Taubes)

Up to diffeomorphism, there is a unique symplectic form on CP^2 of any fixed volume.

Let ω_{FS} and ω_0 embedding with the same volume.

• Step 1: (Gromov) There are diffeomorphisms f so exotic
$\exists$ ω_0-symplectically embedded line $\mathbb{CP}^1$
i.e. a symplectic sphere of degree 1 $\quad S : S \cdot S = 1$
i.e. $\quad [S] = [\mathbb{CP}^1]$

• Step 2: (Taubes) Since $\mathrm{SW}(\mathbb{CP}^2, \Gamma_0) \neq 0$ $\quad c_1(\xi) = PD([\mathbb{CP}^1])$
$\Rightarrow$ embedded symplectic surface in $[\mathbb{CP}^1]$ $\qquad H_2(\mathbb{CP}^2) = \mathbb{Z}$
• C is connected: $C \cdot C \geq 1$ $\qquad \oplus \mathbb{Z}$
$C = C_1 \vee \cdots \vee C_n$ $\quad (C_i) = \pm 1, \ (\pm 1, 0) \ [\omega_{FS}]^2 = 1$
↳ C is a sphere (adjunction formula) $\quad C_i \cdot C_i \geq 1$ $\quad \oplus (\mathbb{Z})^n$

$2g - 2 = C \cdot C + c_1(K) \cdot C \geq 1 \cdot (-3) \cdot 1$, i.e. $-2 = g \geq 0$

Open question: Is there (X, ω) homeomorphic to CP^2 but not diffeomorphic?
↳ like symplectic homology CP^2

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(3) Enlarged and ruled surfaces

Theorem: [Lee, Wieb-Ono] (X, ω) minimal, TFAE
• X is diffeomorphic to ruled
• X admits a metric of PSC
• X admits a symp. form ω with $c_1(\omega) \cdot [\omega] < 0$

Theorem: [Li-Liu, Lalonde-McDuff]
Symplectic structures on a ruled surface are unique up to diffeomorphism and deformation

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④ Taubes's Gromov invariant

• Moduli space of connected embedded J-curves in class $w \in H_2(Y, \mathbb{Z})$
$\dim \mathcal{M}(Y, J, w) = w \cdot w - c_1(K) \cdot w$
• Moduli space of SW monopoles for the spin^c structure ξ, $\mathcal{C}_\xi(\xi) = \mathcal{M}(\xi)$
$\dim \mathcal{M}(Y, \xi) = c_1(\xi) \cdot c_1(\xi) = c_1(\xi^*) - \chi(\xi) =$
$= w \cdot w - c_1(K) \cdot w$

Try some:

Warning: The curves in Taubes's theorem may be disconnected. Let it still work.

→ This leads to a Gromov invariant $\mathrm{Gr} : H_2(X, \mathbb{Z}) \to \mathbb{Z}$

Roughly speaking: $\mathrm{Gr}(w)$ is an algebraic count of (weighted disconnected) embedded J-holomorphic curves in class w.

→ if the expected dim > 0: impose point constraints

→ compactness theorem: Gromov-Hofer: multiply covered has codim ≥ 2

→ if $b^+ > 1$: index problem $\Rightarrow$ consider multiply covered spheres, $S \cdot S = -1$
$\Rightarrow$ negative multiplicity or < 0

→ Need the theorem on Gromov-Witten invariant, although similar.

Theorem: (Taubes) $\qquad \mathrm{SW} = \mathrm{Gr}$ $\qquad$ 5 steps

(X, ω) symplectic, $\quad w \in H_2(X, \mathbb{Z})$ $\qquad L_\xi(w) \in \mathbb{Z}$
$\mathrm{SW}(X, \Gamma_\xi) = \mathrm{Gr}(X, \omega)$ $\qquad \Phi_\xi \downarrow \cong \qquad \downarrow \Phi$
$c_1(\xi) \cdot PD(w)$ $\qquad H^2(X) \xrightarrow{\mathrm{SW}}$
↳ this can be computed topologically

Example: $X = \mathbb{CP}^2$, $\quad \alpha \in [\mathbb{CP}^1]$ $\qquad (w \cdot w \in 1, \ c_1(K) \cdot w = -3)$
$\dim \mathcal{M}^{\mathrm{reg}}(X, J, \alpha) = 1 - (-3) = 4$ $\quad$ (2 complex dimensions)
→ no negative & point constraints
→ In the plane, there is a complex line through any 2 points
$\Rightarrow$ $\mathrm{Gr}(\mathbb{CP}^2, [\mathbb{CP}^1]) = 1 \quad = \quad \mathrm{SW}^{\mathcal{P}}(\mathbb{CP}^2, \xi) = 1$

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⑤ Some ideas of proofs

SW equations on a symplectic manifold (Y, ω)
• compatible $J \in \mathcal{J}(Y, \omega)$ $\qquad W_{can} = A^{0,*}$
• spin^c structure, Γ_{can}, Γ_ξ
• canonical spin^c connection: $A_{can} \Leftrightarrow \Gamma_{can}$
↳ you need to be more careful when $\exists$ no real integrable

Eqn: The eq. becomes
(Notation: $\Phi = \overline{\Phi}$) when

$$\begin{cases} \overline{\partial}_{\Phi} \Psi_1 + \overline{\partial}^*_{\Phi} \Psi_2 = 0 \\ \{ (\Psi_{1,1} + \Psi_{2,2}) + \sqrt{-1} \cdot \Phi \cdot \Psi_1 \\ \ast_1 (\Psi_{1,1} - \Psi_{2,2}) = \Phi \Psi_2 - \Phi \Psi_1 \end{cases}$$

$\Phi \in \Omega^{0,2}(\mathbb{C})$ & $\Omega^{0,*}(\mathbb{C})$
$\Psi \in \Omega^{0,2} \oplus \Omega^{0,0}(X)$

(a) Non-vanishing

Idea: consider the perturbation $\eta : -\overline{\partial}_{can} + \eta \Psi_1 = 0$, Ψ_2 large
$\Longrightarrow$ Many computations, obstacles, ...
$\Rightarrow$ Useful to take $\Psi_2 = \frac{1}{\sqrt{-1}} \Psi_1$, $\quad \Psi_1 = \frac{1}{\sqrt{-1}} \Psi_2$

Conclusion: For the local bundle $L = K \times \mathbb{C}$, for λ large enough, every solution of gauge equivalent to: $\Phi = 0$, $\Psi_2 = \overline{\Psi_1}$, $\Psi_1 = 0$.
$\Rightarrow$ the moduli space is a point (need to check regularity, orientation)

(b) Existence of J-holomorphic curves (hint for exam)

Idea: $\mathrm{SW}(Y, \xi) \neq 0$, then there are solutions for any perturbation η (only there $\eta = 0$ or $\neq 0$)

Usual proposes: The system $\eta \Psi_1$ "reacts to Φ" $\equiv \overline{\partial} \Phi$, Ψ_2 "reacts to Φ" $= 0$
$\rightarrow$ but Ψ_1 cannot happen when $\Phi = 0$ not twisted

Gromov-Witten picture: $\Psi_2 \longrightarrow \overline{\Psi_1}$ almost everywhere, but more complex. Taubes investigate, the behaviour of $\Gamma_\xi = \Psi_1^{-1}(0)$ as $\lambda \to \infty$
Look at limit of embedded submanifolds Γ_ξ converges to J-holomorphic curves.

(3) For the SW

• Gromov - Taubes, appropriate Sobolev completion and on the curve (Sobolev spaces)

Perturb η to a transversality