3d SW invariants.

<u>Setup.</u> M^3 closed oriented Riemannian 3-manifold

$\mathfrak{s} \in Spin^c(M) \neq 0$

<u>Remark.</u> There always exists one; $H_1(M; \mathbb{Z})$.

Associated spinor bundle $S \to M$
(complex rank 2)

$L = \det(S)$ associated line bundle.

$c : T_{\mathbb{C}}^* X \longrightarrow End_0(S)$

A a hermitian connection on L

$\rightarrow D_A$: dirac operator.

SW equations in 3D:

Pairs $(A, \psi) \in \mathcal{A}(L) \times \Gamma(W)$

$$\begin{cases} D_A \psi = 0 \\ c(*F_A + i * \mu) = \psi \otimes \psi^* - \frac{1}{2}|\psi|^2. \end{cases}$$

$\mu \in \Omega^2(M)$ is some closed perturbation.

Remark: you can obtain them from 4D SW equations on $M \times S^1$!

Gauge group $\mathcal{G} = C^\infty(M, S^1)$ acts on solutions via

$$u(A, \psi) = (A + 2u\, du\, u^{-1},\ u \cdot \psi).$$

Moduli space of solutions

$$\mathcal{M}_\mu = \{\text{solutions to the equations}\} / C^\infty(X, S^1).$$

<u>important things:</u>

1. μ generic $\rightsquigarrow$ $\mathcal{M}_\mu$ zero dimensional, compact.

if 3-manifold is compact

$\rightsquigarrow$ inherits a canonical orientation

Count points with signs ; get

$$SW : \mathrm{Spin}^c(M^3) \longrightarrow \mathbb{Z}.$$

Can package these numbers as follows:

give me $\alpha \in H^1(X;\mathbb{Z})$.

$$\underline{SW}(M^3;\alpha) := \sum_{\mathfrak{s} \in Spin^c(M^3)} SW(\mathfrak{s})\, t^{\alpha \smile c_1(\det \mathfrak{s})/2}$$

3-way relationship between 3 concepts:

$$\begin{tikzcd}
\text{3d SW invariants} \arrow[d, leftrightarrow, "\text{Thomas Mark}"'] \arrow[dr, leftrightarrow, "\text{Meng–Taubes, Turaev}"] & \\
S^1\text{-valued Morse theory} \arrow[r, leftrightarrow, "\text{Hutchings–Lee}"'] & \text{Reidemeister torsion}
\end{tikzcd}$$

theorem 1

$$\sum_{\Gamma \in Spin^c(X)} SW(\Gamma)\, t^{\alpha c_1(\det \nabla)/2} = T_{top}(X; [\alpha]).$$

"SW and Torsion"

I&K.

topological invariant of X

Theorem 2

there is an equality

$$T_{top}(X; [\alpha]) = T_{Morse} \cdot \zeta(\alpha).$$

nonTorsion

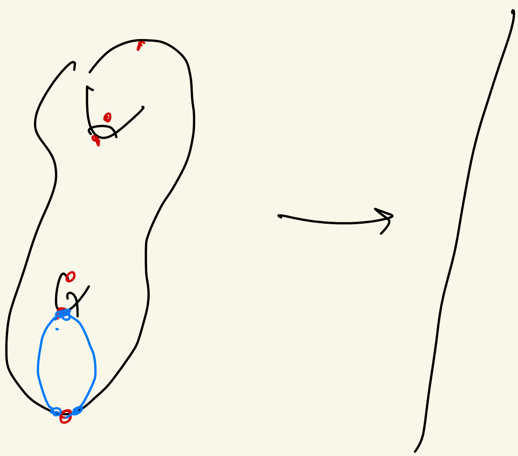
§1. Morse theory and torsions.

Reminder X is a smooth n-manifold.

A Morse function is a smooth function

$$\phi : X \longrightarrow \mathbb{R}$$

with nondegenerate, isolated critical points.



Definition. The Morse complex of a pair $(X, \phi : X \to \mathbb{R})$ is the chain complex

$$(C_i^{Morse}, d) \quad \text{where:}$$

$$C_i^{Morse} = \mathbb{Z}\langle \text{critical points of index } i \rangle.$$

$$\underset{\substack{\uparrow \\ \text{index } i}}{d(x)} := \sum_{y \in Crit_{i-1}} \langle dx, y \rangle\, y$$

counts a signed number of downward flow lines.

<u>theorem</u>.

1. $d^2 = 0$

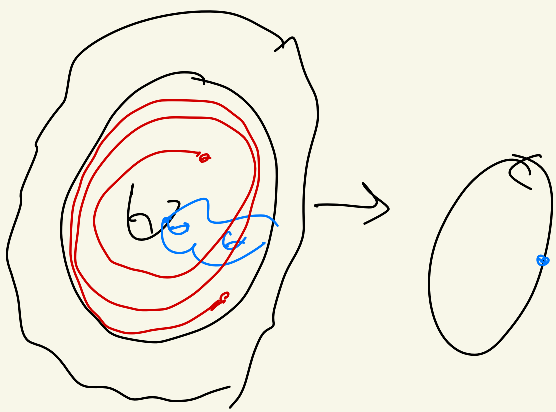
2. $H_i(C_i^{Morse}, d) = H_i(X)$.

S^1-valued Morse theory.

idea: instead of looking at generic maps $X \to \mathbb{R}$, look at maps $X \to S^1$.

<u>Definition</u>: A smooth map $\phi : X \to S^1$ is Morse if the associated map $\tilde{\phi} : \tilde{X} \to \mathbb{R}$ is Morse.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d] \\ X \arrow[r, "\phi"'] & S^1 \end{tikzcd}$$



Morse complex $\rightsquigarrow$ "Novikov complex"

Def. The Novikov complex of $(X, \phi : X \to S^1)$

is the complex $(C_*^{\text{novikov}}, d)$ $S.$

(assume 0 is regular, so that $\Phi^{-1}(0)$ is

a smooth submanifold of X)

$$C_i^{\text{novikov}} = \text{free } \mathbb{Z}((t)) \text{-module generated by } \mathrm{Crit}_i.$$

$$d(x) = \sum_{\substack{x \text{ is index } i \\ y \in \mathrm{Crit}_{i-1}}} \sum_{k=0}^{\infty} \underline{\langle dx, y \rangle}^k t^k y$$

$\langle dx, y\rangle^k$ is the number of flow lines

from x to y crossing S k times.

Theorem. This is a complex: can compute its homology.

Topologically, it computes the following:

1. Pick a cell decomposition of X

2. Lift this to a decomposition of $\tilde{X}$.

Look at $C_i^{\text{cellular}}(\tilde{X})$

This is a $\mathbb{Z}[t, t^{-1}]$ - module:

t acts by shifting cells downwards.

Theorem (Novikov)

$$H_i(C_\cdot^{\text{novikov}}, \phi) = H_i\left(C_\cdot^{\text{cell}}(\tilde{X}) \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}((t))\right).$$

Torsion.

$$\left\{ \begin{array}{l} \text{acyclic chain complex of vector spaces}/k \\ C_n \xrightarrow{\partial} C_{n-1} \to \dots \to C_0 \\ \\ w_i \in \Lambda^{\mathrm{top}} C_i \end{array} \right\} \to \tau \in k^\times$$

$\partial C_i = \mathrm{im}(\partial)$ on C_i

$$0 \to \underbrace{C_n}_{v_{i+1}} \to \underbrace{C_{n-1}}_{w_i} \to \underbrace{\partial C_{n-1}}_{v_i} \to 0$$

$$0 \to \partial C_{i+1} \to \underline{C_i} \to \partial C_i \to 0$$

$i = 2, \dots, n-2$

$$0 \to \underline{\partial C_2} \to C_1 \to \underline{C_0} \to 0$$

1. Pick arbitrarily volume elements $v_i \in \partial C_i$.

2. Consider:

$$\tau_i = \frac{v_{i+1}\, v_i}{w_i} \quad \text{for } i = 2, \ldots, n-2.$$

$$\tau_{n-1} = \frac{w_n \cdot v_{n-1}}{w_{n-1}}$$

$$\tau_1 = \frac{v_2\, w_0}{w_1}$$

Def. The torsion of this data is

$$\tau(C_\bullet, \{w_i\}) = \prod_{i=1}^{n-1} \tau_i^{(-1)^{i+1}}$$

$$\in k^\times.$$

Topological torsion

X manifold, $\phi : X \to S^1$ Morse function.

$$C_i^{cell}(\tilde{X}) \otimes \mathbb{Q}((t)).$$

choice of lift for each i-cell.

induced volume form is not unique,

but it is unique up to:

1. ± 1
2. Powers of t (shift cells)

Def. if the complex is acyclic, then.

$$T_{top}(X, \phi) := \mathcal{T}\left(C_i^{cell}(\tilde{X}) \otimes \mathbb{Q}((t)),\right.$$

picking lifts of cells).

$$\in \frac{\mathbb{Q}^{\times}((t))}{\pm t^k}$$

Remark: this only depends on ϕ by the its class in $H^1(X;\mathbb{Z})$.

Classical example

$K \subset S^3$, let X_K be the 0-surgery on K.

if α generates $H^1(X_K;\mathbb{Z})$ then

$$T_{top}(X,\alpha) = \frac{\text{Alexander}(K)}{(1-t^2)} \in \mathbb{Z}(t,t^{-1})/\pm t^k$$

$\overline{S}^1$-valued morse theory.

Can also compute "Morse" torsion:

Torsion of $C_i^{\text{novikov}}(X, \phi) \otimes \mathbb{Q}((t))$

Basis: critical points.

Def.

$$T_{\text{morse}}(X, \phi) = \tau\left(C_i^{\text{novikov}}(X, \phi) \otimes \mathbb{Q}((t)),\ \text{critical points} \right)$$

$$\in \mathbb{Q}((t)) / \pm 1$$

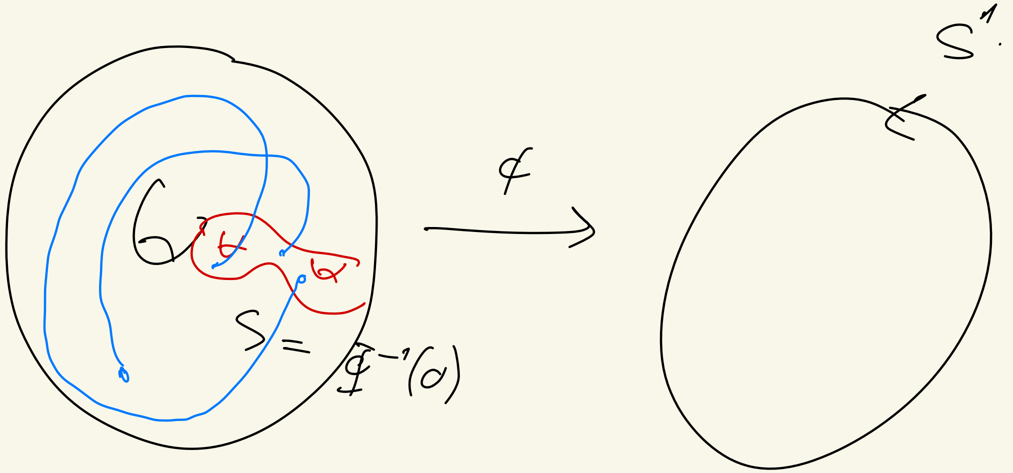
Question. Do the torsions match?

$$T_{\text{top}}(X, \phi) \cong T_{\text{morse}}(X, \phi) ?$$

$$\mathbb{Q}((t)) / \pm t^k$$

Answer is : no! There is a connection from

"Zeta function of ϕ".



Get a partial map

$F : S \nrightarrow S.$

Def. $\zeta(\Phi) = \exp\left(\sum_{k \geq 1} \mathrm{Fix}(F^k) \frac{t^k}{k} \right)$

$\in \mathbb{Q}[[t]].$

For any manifold X and Φ Morse function:

Theorem.

$$T_{\text{morse}} \cdot \zeta(\phi) = T_{\text{top}} \quad \text{in } \mathbb{Q}((t))$$

up to $\pm t^k$.

|| 3-manifold case.

$\underline{SW}(t, [\phi])$.

§. Relation between Morse functions and SW invariants:

Donaldson: Can construct a TQFT out of SW on 3-manifolds.

Suppose $X = \Sigma \times \mathbb{R}$ Σ Riemann surface.

Give me a Spin^c-structure; $S \to X$
spinor bundle.

$dt \curvearrowright S$; $(dt)^2 = -1$.

$\rightsquigarrow$ splits S into two eigenbundles

$$S = F \oplus E$$
$$\begin{array}{cc} \uparrow & \uparrow \\ i & -i. \end{array}$$

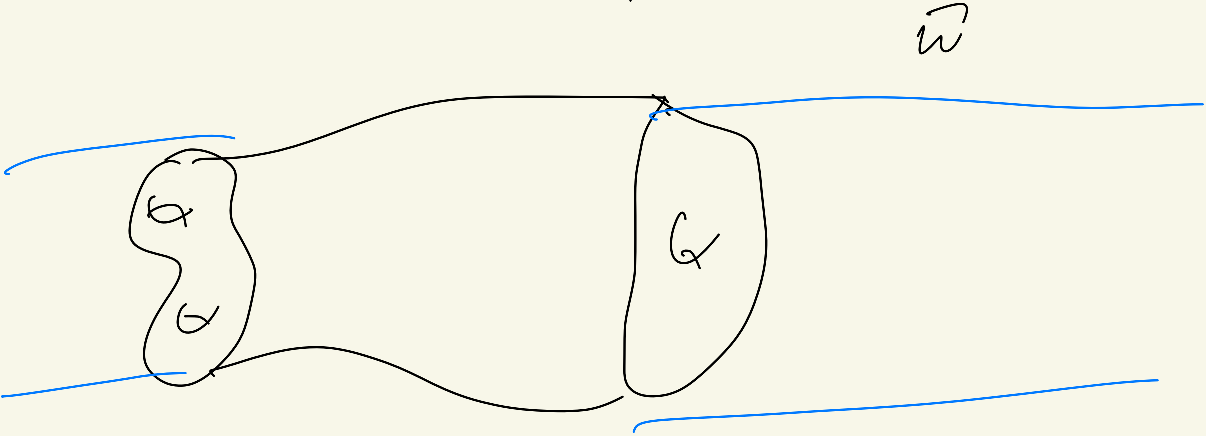
Write down SW equations on this:

they reduce to "vortex equations"
on Σ for E.

Vortex equations are well-understood:
Moduli space of solutions is $\mathrm{Sym}^{\deg(E)}(\Sigma)$.

Give me a cobordism

$$W : S_- \longrightarrow S_+$$



Pick a structure on $\hat{W}$; and an instanton.

Solutions will look like solutions to Morse

vector gradient near the boundary.

$$M(\hat{W}, \Gamma) \xrightarrow{\rho_{-,+}} M(S_{\pm})$$

which take the solutions to the surfaces.

$$\rho : M(\vec{w}) \xrightarrow{\rho_-, \rho_+} M(S_+) \times M(S_-)$$

has a fundamental class $[M(\vec{w})]$.

consider

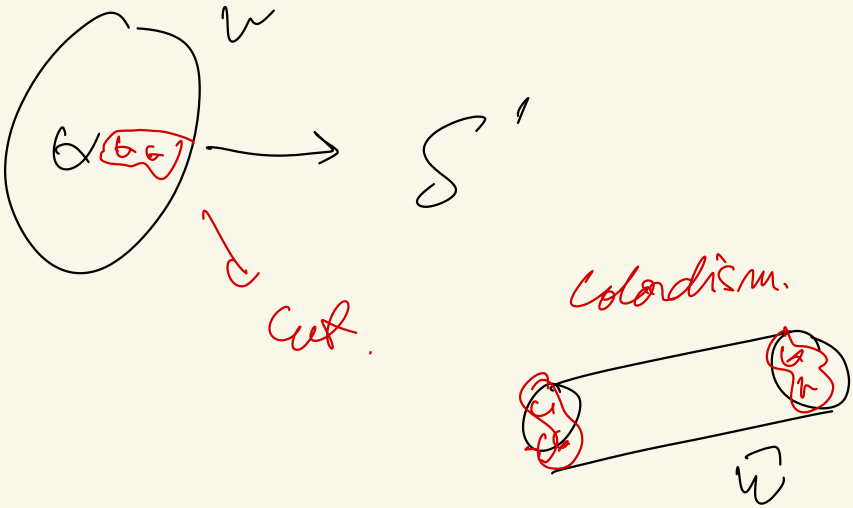
$$\rho_*[M(\vec{w})] : H_*(M(S_+)) \otimes H_*(S_-))$$

$$= \hom(H^*(M(S_-)), H^*(M(S_+))).$$

given a cobordism

$\to$ get a map $H^*(M(S_-)) \to H^*(M(S_+))$.

What kind of invariant on 3-manifolds?



Theorem (Donaldson).

$$\mathrm{Tr}(\mathrm{map}_{\hat{W}}) = \sum_{\substack{\text{spin structures on } \hat{W} \\ \text{that glue to a spin} \\ \text{structure on}}} SW(W, \mathfrak{s}).$$