SW invariants of Kähler surfaces

Plan:
- canonical spin^c structure
- Enriques–Kodaira classification
- $SW(X, \Gamma_{can})$, $SW(X, \Gamma^*_{can}) \neq 0$
- ...

← review of Kähler geometry along the way

① Canonical spin^c structure

1.1) Linear case

So far: V f.d. oriented real inner product space $g(\cdot,\cdot)$ $(SO(n))$
↳ spin^c structures $\Gamma: V \to End(W)$

What if V is already a complex v.s.? $(U(k))$ $n=2k$

Def: A complex structure on V: $J: V \to V$ s.t. $J^2 = -id$
- compatible w. the inner product: $g(Jv, Jw) = g(v,w) \to J$ orthogonal
- compatible orientations

(V, J, g) is a Hermitian vector space.
- $\omega(v,w) = g(Jv, w) \to$ non-deg. antisym. bilinear form
- We get a Hermitian inner product $\langle v,w\rangle := g(v,w) + i\omega(v,w)$

$\langle \cdot,\cdot\rangle : V \times V \to \mathbb{C}$
antilinear, linear

Example: $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $z_j = x_j + iy_j$ $\quad J(x,y) = (y,-x)$

$\langle v,v'\rangle = \sum_j \bar{z}_j z_j'$ $\qquad \omega = \sum_j dx_j \wedge dy_j = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$

Forms: $\Lambda^1 V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$ real-valued 1-forms
$\Lambda^1 V^* \otimes \mathbb{C} = Hom_{\mathbb{R}}(V, \mathbb{C})$ complex-valued 1-forms

Splitting: $\Lambda^1 V^* \otimes \mathbb{C} = \Lambda^{1,0} V^* \oplus \Lambda^{0,1} V^*$
$Hom_{\mathbb{C}}(V, \mathbb{C}) \oplus Hom_{\mathbb{C}}(\bar{V}, \mathbb{C})$
$dz_j \qquad d\bar{z}_j$

More generally: $\Lambda^k V^* \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} V^*$

Example: $n=2$ $\quad \Lambda^2 V^* \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$
$dz_1 \wedge dz_2 \quad \vdots \quad d\bar{z}_1 \wedge d\bar{z}_2$

- A Hermitian v.s. has a canonical spin^c structure.

Conceptually:

$$\begin{tikzcd} & Spin^c(2n) \arrow[d] \\ U(n) \arrow[ur, dashed, "\exists !"] \arrow[r, "A \mapsto (A, \det A)"'] & SO(2n) \times S^1 \end{tikzcd}$$

$U(n) \subseteq SO(2n)$

Explicitly: $\Gamma_{can}: V \to End(W_{can})$, $W_{can} = \Lambda^{0,*} V^* = \bigoplus_{k=0}^{n} \Lambda^{0,k} V^*$ ← complex anti-linear forms

$\Gamma_{can}(v)\tau = \frac{1}{\sqrt{2}} v^\flat \wedge \tau - \sqrt{2}\, \iota_v \tau$, $\qquad v^\flat = \langle \cdot, v\rangle \in \Lambda^{0,1} V^*$
$\tau \in W_{can}$

Splitting: $W_{can} = W^+_{can} \oplus W^-_{can}$, $\quad W^+_{can} = \Lambda^{0,even} V^*$
$W^-_{can} = \Lambda^{0,odd} V^*$.

1.2) Over a manifold

Before: (X, g) oriented Riemannian manifold of dim. $2n$ $(SO(2n))$

Def: An almost complex structure on X is $J: TX \to TX$ s.t. $J^2 = -id$.
- compatible with the metric and the orientation

(X, g, J) almost Hermitian manifold

- $\omega(v,w) := g(Jv, w)$, $\omega \in \Omega^2(X)$ is a non-deg. 2-form
- We get a Hermitian metric $\langle \cdot,\cdot\rangle = g + i\omega$ $\quad \langle\cdot,\cdot\rangle: TX \otimes TX \to \mathbb{C}$
- An a.c.s. J is integrable if it arises from complex coord. on X with holomorphic transition maps.
$\iff$ the Nijenhuis tensor N_J vanishes
- A Hermitian manifold is a triple (X, J, g) where
  → (X,g) Riemannian
  → J is an integrable a.c.s.
  → J is orthogonal wrt g

We say (X, J, g) is Kähler if the 2-form ω is closed, $d\omega = 0$.
$\Rightarrow \omega$ is a symplectic form

Basic example: $\mathbb{CP}^n$ with the Fubini–Study metric $\quad \mathbb{CP}^n = \frac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^*}$

- Differential forms: $\Omega^k(X;\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$

The exterior diff. $d: \Omega^k \to \Omega^{k+1}$ also induces
$\partial: \Omega^{p,q} \to \Omega^{p+1,q}$, $\quad \bar\partial: \Omega^{p,q} \to \Omega^{p,q+1}$.

Notation: Ω^k is the global sections of the bundle $\Lambda^k T^*X$.

Def: The canonical bundle of X is the complex line bundle $K = \Lambda^{n,0} T^*X$.
← holomorphic top form
anticanonical bundle $K^* = \Lambda^{0,n} T^*X$

Note: $c_1(K) = -c_1(X) = -c_1(TX)$, $\quad c_1(K^*) = c_1(X)$

- Again, we have a canonical spin^c structure on X
$\Gamma_{can}: TX \to End(W_{can})$, $\quad W_{can} = \Lambda^{0,*} T^*X = W^+_{can} \oplus W^-_{can}$

The determinant line bundle $L_{\Gamma_{can}} \cong K^*$.

Recall: Any other spin^c structure on X is of the form
$\Gamma_E: TX \to End(W_E)$, $\quad W_E = W \otimes E$, $E \to X$ Hermitian line bundle.
$L_{\Gamma_E} \cong K^* \otimes E^{\otimes 2}$

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② Kähler surfaces: examples and classification

First example: $\mathbb{CP}^2$ and its blow-ups $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ (rational surfaces)

Recall from Shing: $X \xrightarrow{\text{blow-up}} \tilde{X} \underset{diff.}{\cong} X \# \overline{\mathbb{CP}^2}$

this construction replaces a point w/ an exceptional divisor $E \cong \mathbb{CP}^1$

E is an embedded holomorphic sphere of self-intersection -1

Def: X is minimal if it contains no exceptional divisors.
↳ there's a "classification" of minimal Kähler surfaces

Def: X is a ruled surface if there is a holomorphic fibration $\pi: X \to \Sigma$ over a Riemann surface Σ, with fibre $\mathbb{CP}^1$.

↳ (diff.) orientable S^2-bundle over a closed orientable surface Σ_g

diff. types: $S^2 \times \Sigma_g$, $\quad S^2 \tilde{\times} \Sigma_g$

When $\Sigma = \mathbb{CP}^1$ we get the Hirzebruch surfaces (both rational and ruled)
↘ e.g. $\mathbb{CP}^1 \times \mathbb{CP}^1$

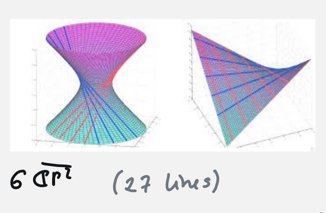
If $g \geq 1$, we get the irrational ruled surfaces.

- Hypersurfaces in $\mathbb{CP}^3$:

$f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ homogeneous polynomial of deg. d
↳ defines a projective variety $Z(f) = \{[z_0 : \cdots : z_3] \in \mathbb{CP}^3 : f(z) = 0\}$
assume it is smooth

Fact: They are all diffeomorphic for f of fixed degree.
$X_d = \{z_0^d + \cdots + z_3^d = 0\} \subset \mathbb{CP}^3$.

All X_d are simply connected.



- $d=1$: X_1 is a hyperplane $\cong \mathbb{CP}^2$
- $d=2$: $X_2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$
- $d=3$: cubic surface $X_3 \cong \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$ (27 lines)
- $d=4$: X_4 is a K3 surface — Kummer, Kodaira, Kähler, K2 (Weil)

Def: A K3 surface is a simply connected Kähler surface with trivial canonical bundle.
↳ they are all diffeo. $\quad \chi = 24$, $\sigma = -16 \to b_2 = 22$, $b^+ = 3$, $b^- = 19$
$Q = 2(-E_8) \oplus 3H$

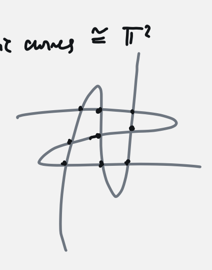
- $d \geq 5$: X_d is a "surface of general type"

Fact: Q_{X_d} is indefinite $(d \geq 2)$
even $\iff d$ is even

$\Rightarrow d$ odd, $\Rightarrow Q_{X_d}$ is diagonalisable $\ell(1) \oplus m(-1)$ $\qquad$ **Q:** $X_d \cong \ell\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$?
No.

Def: X is an elliptic surface if there is a holomorphic map $\pi: X \to \Sigma$ to a Riem. su. Σ s.t. the fibres are elliptic curves.
↳ all but finitely of the fibres are smooth elliptic curves $\cong \mathbb{T}^2$

Examples:



- rational elliptic surface $V_{-1} \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ $\quad E(1)$
- more generally: $V_k = V_{k-1} \#_{\mathbb{T}^2} V_{-1}$ fibre sum
- e.g. V_0 is a K3 surface
V_k $k \geq 1$ "proper" elliptic surface.

Classification:

X Kähler surface, K canonical bundle, $H^0(X, K)$, $H^0(X, K^{\otimes m})$? ← holomorphic sections

$\dim H^0(X, K^{\otimes m}) \sim m^d$ for some $d \to$ Kodaira dimension $Kod(X)$

$Kod(X) = \limsup_{m\to\infty} \frac{\log \dim H^0(X, K^{\otimes m})}{\log m}$. → birational invariant

Theorem: The Kodaira dim. of a Kähler surface is $-\infty$, 0, 1 or 2.

Assume X is minimal.

- $Kod(X) = -\infty$: X is $\mathbb{CP}^2$ or a ruled surface
↳ they always have Kähler metrics w/ PSC ($\Leftarrow$!) e.g. LeBrun
- $Kod(X) = 0$: then X is a finite quotient of a K3 surface or of $\mathbb{T}^4$ ($b^+ = 3$)
Moreover, $c_1(K)$ is torsion.
  → $K3/\mathbb{Z}_2$: Enriques surfaces
  → $\mathbb{T}^4/\mathbb{Z}_m$: hyperelliptic surfaces $(m = 2, 3, 4, 6)$ $\quad \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$
  ↳ $b^+ = 1$
- $Kod(X) = 1$: then X is an elliptic surface.
Moreover, $c_1(K)$ is not torsion, but $c_1(K)^2 = 0$
- $Kod(X) = 2$: surface of general type, $c_1(K) \cdot c_1(K) > 0$.

③ SW invariants of Kähler surfaces

X, $\Gamma_{can}: TX \to End(W_{can})$ $\quad W^+_{can} = \Lambda^{0,0} \oplus \Lambda^{0,2}$, $\quad W^-_{can} = \Lambda^{0,1}$
$L_{\Gamma_{can}} = K^* = \Lambda^{0,2}$

- There is a canonical spin^c connection: the Levi-Civita connection on forms.

In general: $A \in \mathcal{A}(\Gamma_{can})$ is $A = A_{can} + B$, $\quad B \in \Omega^1(X, i\mathbb{R})$.

- More generally: $\Gamma_E: TX \to End(W_E)$
$W_E^+ = (\Lambda^{0,0} \oplus \Lambda^{0,2}) \otimes E$, $\quad L_{\Gamma_E} = K^* \otimes E^{\otimes 2}$

A spin-c connection $A \in \mathcal{A}(\Gamma_E)$ is $A = A_{can} + B$, $\quad B \in \mathcal{A}(E)$.

B Hermitian connection on E

- Dirac operator: $D_A: \Omega^{0,even}(X, E) \to \Omega^{0,odd}(X, E)$
$D_A = \sqrt{2}(\bar\partial_B + \bar\partial_B^*)$

The connection B induces $d_B: \Omega^k(X, E) \to \Omega^{k+1}(X, E)$
$\bar\partial_B: \Omega^{p,q}(X, E) \to \Omega^{p,q+1}(X, E)$

Prop: X Kähler surface with spin^c structure Γ_E
$\Phi = (\varphi_0, \varphi_2) \in \Omega^{0,0}(X, E) \oplus \Omega^{0,2}(X, E)$, $\quad A = A_{can} + B$, $B \in \mathcal{A}(E)$

perturbation $\eta \in \Omega^{2,+}(X, i\mathbb{R})$

The SW equations become:

$D_A \Phi = 0$
$F_{A_{can}} + F_B + \eta = \sigma^{-1}((\Phi\Phi^*)_0)$

$$\begin{cases} \bar\partial_B \varphi_0 + \bar\partial_B^* \varphi_2 = 0 & (1) \\ 2(F_B + \eta)^{0,2} = \bar\varphi_0 \varphi_2 & (2) \\ 4i(F_{A_{can}} + F_B + \eta)_\omega = |\varphi_2|^2 - |\varphi_0|^2 \end{cases}$$

$(\tau \in \Omega^2, \quad \tau \wedge \omega = \tau_\omega\, \omega \wedge \omega)$
$\tau_\omega: X \to \mathbb{C}$

Prop: If $\eta \in \Omega^{1,1} \cap \Omega^{2,+}$, then any solution must satisfy $\varphi_0 = 0$ or $\varphi_2 = 0$.

$\bar\partial_B(1)$ $\quad \bar\partial_B \bar\partial_B^* \varphi_2 = -\bar\partial_B^2 \varphi_0 \stackrel{!}{=} -F_B^{0,2} \varphi_0 = -\frac{1}{2}|\varphi_0|^2 \varphi_2$

L^2 inner product $\longrightarrow$ $\int_X \left(|\bar\partial_B^* \varphi_2|^2 + \frac{1}{2}|\varphi_0|^2 |\varphi_2|^2\right) dvol_g = 0$

e.g. $\varphi_2 = 0$ $(\eta = 0)$ $\longrightarrow$
$$\begin{cases} \bar\partial_B \varphi_0 = 0 \\ F_B^{0,2} = 0 \\ 4i(F_B)_\omega = \tau - |\varphi_0|^2 \end{cases}$$
$\tau = -4i(F_{A_{can}})_\omega$

Vortex equations

Recall: dual spin^c structures $\Gamma_E: TX \to End(W_E)$, $\bar\Gamma_E \cdots \bar W_E$
$\bar\Gamma_E = \Gamma_{K \otimes E^*} \longrightarrow c_1(\bar\Gamma_E) = -c_1(\Gamma_E)$
e.g. E trivial. $\bar\Gamma_{can} = \Gamma_K$, $\quad L_{\bar\Gamma_{can}} = K^* \otimes K^2 = K$

Theorem: For any Kähler surface X, $SW^+(X, \Gamma_{can}) = 1$
$SW^+(X, \bar\Gamma_{can}) = (-1)^{\frac{\sigma+\chi}{4}}$.
$b^+ = 1$

Moreover if $SW(X, \Gamma_E) \neq 0$, then $0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega]$.
with equality iff $E = 0$ or $E = K$ $\quad (b^+ = 1?)$

Corollary: If $b^+ > 1$ and $c_1(K)$ is torsion, then $\pm c_1(K)$ are the only basic classes.
↳ surfaces of $Kod(X) = 0$: K3 surface and $\mathbb{T}^4$
↳ $(b^+ = 1?)$

Corollary: hypersurface X_d for $d \geq 5$ odd
$Q_{X_d} = \ell(1) \oplus m(-1)$ $\quad \ell \geq 2$
- Freedman $\Rightarrow X_d$ is homeomorphic to $\ell\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2} = X_d'$
- SW $\Rightarrow X_d$ has $\neq 0$ SW inv.
X_d' has all SW $= 0$ (connected sum vanishing)

Thus X_d and X_d' are not diffeomorphic.