§0 – Outline

- Recap on intersection form
- Comparing smooth vs topological 4-mflds
  - ↳ Existence of non-smoothable 4-mflds.
- Sketch proof of Donaldson's diag. thm.
- Some more?

§1 – Intersection forms

X a closed, connected, oriented 4-mfld (TOP / smooth).

↳ Intersection form $Q_X : \frac{H_2(X)}{\text{torsion}} \times \frac{H_2(X)}{\text{torsion}} \longrightarrow \mathbb{Z}$.

__Key facts__

- Symmetric, bilinear form on $\mathbb{Z}^{b_2}$.
- Unimodular – any matrix rep. has $\det = \pm 1$. (Uses $\partial X = \emptyset$).

__Key-properties__

- Parity: Q_X is __even__ if $Q_X(\alpha,\alpha) \equiv 0 \bmod 2$ $\forall \alpha \in H_2(X)/\text{tors}$
  " __odd__ otherwise.

  Note: diagonalisable forms are odd (or trivial)
- Definiteness: Q_X is __positive__ (__negative__) __definite__ if $Q_X(\alpha,\alpha) > 0$ (<0) $\forall \alpha$.
  " __indefinite__ otherwise.
- Signature: $\sigma(X) = \sigma(Q_X)$

__Key questions__ ① Which forms can be realized as intersection forms of smooth / top 4-mflds? ← geometric

② What forms are there? ← algebraic

§2 – Classification of intersection form

__Thm__ (Hasse-Minkowski) Let Q be a unim. sym. bil. form over $\mathbb{Z}^n$.

① If Q is odd & indefinite, then it is diagonalisable, so $Q \cong \ell(1) \oplus m(-1)$

② If Q is even & indefinite, then $Q \cong \ell E_8 \oplus n H$

E_8: 8×8, mostly 2 on diagonal, H: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In particular, indefinite forms are classified by parity, signature, & rank.

↳ The definite forms are hard.

__Thm__ (Freedman) Every unim. sym. bil. form is the intersection form of some simply-connected 4-manifold.

↳ If the form is even, ! up to homeo.

↳ " odd, ∃ exactly two s-c 4-mflds w/ that intersection form

§2.1 – Characteristic elements

We call $w \in H_2(X)/\text{tors}$ __characteristic__ if $Q_X(w,x) = Q_X(x,x) \bmod 2$ $\forall x$.

↳ Recall: X is spin if $w_2(X) = 0 \in H^2(X;\mathbb{Z}/2)$.

(Wu) $w_2(X)$ is characterised by $\langle w_2, \alpha\rangle = \overline{Q_X(\alpha,\alpha)}$ $\forall \alpha \in H_2(X;\mathbb{Z}/2)$

↳ Recall: There is a map

$$\{\text{spin}^c \text{ structures on } X\} \longrightarrow \{c \in H^2(X) \text{ which are integral lifts of } w_2\}$$
$$(W,\Gamma) \longmapsto c_1(L_\Gamma)$$

which is a bijection if $H_1(X) = 0$, & is always surjective

↳ If w is characteristic $\overset{PD}{\rightsquigarrow}$ $c \in H^2(X)$ which is an integral lift of w_2

$c^2 = Q_X(w,w) \in \mathbb{Z}$.

$\rightsquigarrow$ spin^c structure on X, $c_1(L_\Gamma) = c$.

↳ __Rmk__ Q_X is even ⟺ 0 is characteristic.

__Prop__ ① Q_X has a characteristic element (equiv, X admits a spin^c structure)

② $Q_X(w,w)$ for characteristic w is well-defined mod 8.

$w' = w + 2x \longrightarrow Q_X(w',w') = Q_X(w,w) + 4\big(\underbrace{Q_X(w,x) + Q_X(x,x)}_{\text{even}}\big)$

③ $\sigma(X) \equiv Q_X(w,w) \bmod 8$.

$Q_X' = \underset{w}{Q_X} \oplus \underset{\alpha}{(1)} \oplus \underset{\beta}{(-1)} \Rightarrow Q_X'$ odd & indefinite

$\Rightarrow Q_X' \cong \ell(1) \oplus m(-1)$.

$Q_X'(w',w') = \ell - m = \sigma(Q_X') = \sigma(Q_X)$.

$\parallel \bmod 8$

$Q_X(w+\alpha+\beta, w+\alpha+\beta) = Q_X(w,w)$.

④ If Q_X is even, then $\sigma(X) \equiv 0 \bmod 8$. ⚠

§3 – Comparison to the smooth case

__Thm__ (Donaldson, Fintushel-Stern, Furuta, Donaldson)

If X is smooth, definite, then Q_X is diagonalisable, so if X is simply-connected, X is homeo. to $\#^p \mathbb{CP}^2$ or $\#^q \overline{\mathbb{CP}^2}$.

↳ E_8 is __not__ the int. form of a smooth 4-mfld.

↳ All smooth spin 4-mflds have indefinite intersection form.

__Thm__ (Rochlin) If X is smooth & spin, then $\sigma(X) \equiv 0 \bmod 16$.

__Thm__ (Furuta) If X is smooth & spin, $Q_X = 2kE_8 \oplus mH$, & $m \geq 2|k| + 1$.

↳ Conjecture: $m \geq 3|k|$ (true for $k \leq 3$).

$\Updownarrow$

$$\frac{b_2}{|\sigma(X)|} \geq \frac{11}{8}.$$

§4 – Sketch Proof of Donaldson's Theorem

__Thm__ (Elkies) A definite unim. sym. bil. form is diagonalisable iff ∄ characteristic w s.t. $|Q(w,w)| \leq \operatorname{rank}(Q) - 8$.

__Thm__ (Kronheimer) X smooth, definite. Then every characteristic w has $|Q_X(w,w)| \geq b_2$.

__Proof__ ① Argue by contradiction, & assume X is negative definite. Suppose there exists some char. w s.t. $-Q_X(w,w) < b_2 = -\sigma(X)$,

$\exists k \geq 1$ s.t. $Q_X(w,w) + b_2 = 8k$.

② There is some spin^c structure on X s.t. $c := c_1(L_\Gamma) \in H^2(X)$ satisfies $c^2 + b_2 = 8k$. (ie $c^2 = Q_X(w,w)$)

③ Recall we have moduli spaces

$$\widetilde{\mathcal{M}}(X,\Gamma,g,\eta) = \widetilde{\mathcal{M}}(\eta) := \{(A,\Phi) \mid D_A\Phi = 0,\ F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0),\ d^*(A_0 - A) = 0\}$$

A_0: fixed, arbitrary

$$\mathcal{M}(\eta) := \widetilde{\mathcal{M}}(\eta)/\mathcal{G}_0 \qquad (1 \to S^1 \to \mathcal{G}_0 \to H^1(X) \to 1)$$

For generic η, $\widetilde{\mathcal{M}}(\eta)$ is a smooth, compact manifold of dim.

$$\frac{c^2 - 2\chi(X) - 3\sigma(X)}{4} + 1 = \frac{c^2 - 4 + 4b_1 + b_2}{4} + 1 = 2k + b_1.$$

$\mathcal{M}(\eta)$ has singularities at $[A,0]$.

✪ ASSUME $b_1 = 0$ ✪

④ We want to show that $\mathcal{M}(\eta)$ has only 1 singularity & it's a cone on $\mathbb{CP}^{k-1}$.

↳ If $(A,0), (A',0) \in \widetilde{\mathcal{M}}(\eta)$, then $F_A^+ = F_{A'}^+ = -\eta \Leftrightarrow F_A^+ - F_{A'}^+ = 0$.

$\Leftrightarrow F_A - F_{A'}$ is anti-self dual, ie $*(F_A - F_{A'}) = -(F_A - F_{A'})$

$F_A - F_{A'}$ is closed $\Rightarrow F_A - F_{A'}$ is harmonic (Sking)

↳ $[F_A] = [F_{A'}] = 2\pi i\, c_1(L_\Gamma) \Rightarrow F_A - F_{A'} = 0$.

So $A - A'$ by a closed 1-form.

By assumption, $\exists f : X \to \mathbb{R}$ s.t. $A - A' = i\,df$.

$\rightsquigarrow g : X \to S^1$ by $g(x) = e^{if(x)/2}$ takes A to A'.

$g^{-1}dg = i\,df$.

$g \cdot (A,\Phi) = (g^{-1}A + g^{-1}dg,\ g^{-1}\cdot\Phi) = (A',0)$ (up to factors of 2)

↳ __Upshot:__ at most one singularity $[A,0]$.

↳ __Recall__ codimension of Γ^+-wall is $b^+ = 0$.

$\Rightarrow \mathcal{M}(\eta)$ has at least one singularity.

$\Big\}$ $[A,0]$ unique singularity.

↳ ∃ central extension $1 \to S^1 \to \mathcal{G}_0 \to H^1(X;\mathbb{Z}) \to 1$.

S^1 acts by $e^{i\theta}\cdot(A,\Phi) = (A, e^{i\theta}\cdot\Phi)$

By assumption, $b_1 = 0 \Rightarrow H^1(X;\mathbb{Z}) = 0$. $\in \widetilde{\mathcal{M}}(\eta)$

↳ On a small $(2k-1)$-sphere around $(A,0)$, $S^1 = \mathcal{G}_0$ acts by complex multiplication

$\Rightarrow S^{2k-1}/\mathcal{G}_0 \cong \mathbb{CP}^{k-1}$.

⑤ Look at the exterior of the singularity in $\mathcal{M}(\eta)$, call it W.

W is a compact, ori, $(2k-1)$-mfld with $\partial W = \mathbb{CP}^{k-1}$.

- If k is odd, contradiction, $\mathbb{CP}^{2n}$ doesn't bound.
- In general, same idea – Hopf fibration doesn't extend over W.

↳ $\widetilde{\mathcal{M}}(\eta)\setminus\mathring{?}$ is an - - - - - S^1-bundle over W which restricts to the tautological bundle over $\partial W = \mathbb{CP}^{k-1}$.

⚠ Impossible, by calculating Steifel–Whitney numbers.