§1 - Clifford Multiplication

* Let V be oriented, real inner product space, $\dim V = 2n$ or $2n+1$.

$$C(V) = TV/\langle v\otimes v + |v|^2\rangle, \qquad \mathbb{C}^*(V) = C(V)\otimes\mathbb{C}$$

$$S_{Pin}(V) := \{x_1\cdots x_k \mid x_i\in V,\ |x_i|=1\} \subseteq C(V)$$

$$S_{Pin^c}(V) = \langle S, S_{Pin}(V)\rangle = \{s\otimes z \mid \begin{smallmatrix}s\in S_{Pin}(V)\\ z\in S^1\end{smallmatrix}\} \subseteq \mathbb{C}^*(V)$$

$$\left.\begin{array}{ccccccc} 1\to\mathbb{Z}/2 & \to & S_{Pin}(V) & \to & SO(V) & \to 1 \\ 1\to S^1 & \to & S_{Pin^c}(V) & \xrightarrow{\cong} & SO(V) & \to 1 \\ 1\to S_{Pin}(V) & \to & S_{Pin^c}(V) & \xrightarrow{\xi} & S^1 & \to 1 \end{array}\right\} \text{exact.}$$

* Let W be a 2^n-dim'l complex Hermitian space. Then

$$\Rightarrow \quad C^*(V) \cong \begin{cases} End(W) & \text{if } \dim V = 2n \\ End(W)\oplus End(W) & \text{if } \dim V = 2n+1 \end{cases}$$

$\rightsquigarrow$ representations $S_{Pin^c}(V) \longrightarrow Aut(W)$. (*)

$$\begin{array}{ccc} S_{Pin^c}(V) & \longrightarrow & Aut(W) \\ \downarrow & & \downarrow \\ C^*(V) & \longrightarrow & End(W) \end{array}$$

Clifford multiplication

Def A $Spin^c$-structure on V is linear map $\Gamma: V \to End(W)$ s.t.

$$\left.\begin{array}{l} \Gamma(v)^* = -\Gamma(v), \quad \Gamma(v)^*\cdot\Gamma(v) = |v|^2\,\mathbb{1}_W \\ \text{& if } \dim V = 2n+1, \quad \Gamma^*(e_{2n+1})\cdots\Gamma(e_1) = (-1)^n\,\mathbb{1}_W \end{array}\right\} (\dagger)$$

anticommutation

Rmk $\sigma\cdot(\Gamma(v)w) = (\sigma\cdot\Gamma(v))(\sigma\cdot w)$

Vocab Let S be a 2^{2n} dim'l quaternionic symplectic space, i.e. 2^{2n} dim'l real inner product space, with orthogonal anticommuting complex structure $I, J: S\to S$.

Then:

$$C(V) \cong \begin{cases} End_{\mathbb{H}}(S) & \text{if } \dim V = 2 \text{ or } 4 \text{ mod } 8 \\ End_{\mathbb{H}}(S)\oplus End_{\mathbb{H}}(S) & \text{if } \dim V \equiv 3 \text{ mod } 8 \end{cases}$$

Def A $Spin^c$-structure on V is a $\mathbb{R}$-linear map $\Gamma: V \to End_{\mathbb{C}}(S)$ satisfying $(\dagger)$ & commuting with I & J.

Rmk A Spin map $V^n \to X$ is s.t. $\forall v,w$ $\langle \varphi_* v, \varphi_* w\rangle = \langle v, w\rangle$

Riemannian metric

Def ① A $Spin^c$-structure on $V \hookrightarrow E \xrightarrow{\pi} X$ is a principal $S^1_{Spin^c}$-bundle $P \to X$ & a choice of isomorphism $P \times_{S^1} SO(V) \cong SO(E)$.

$\downarrow$ $P\times_{S^1} SO(V)$

$(p, A) \sim (p\cdot r^{-1}, \lambda(r)A)$

② A $Spin^c$-structure on $V \hookrightarrow E \xrightarrow{\pi} X$ is a pair (W, Γ), where $W \to X$ is a Hermitian vector bundle over X of rank 2^n & $\Gamma: E \to End(W)$ is a bundle morphism satisfying $(\dagger)$ in each fiber.

$$P_{Spin^c} = \left\{ (x, \Delta, \mathcal{I}) \;\middle|\; \begin{array}{ll} x\in X & \mathcal{I}: V_x \xrightarrow{\cong} W_x, \\ \Delta: V_x \xrightarrow{\cong} E_x & \text{unitary} \\ \circ\ \mathcal{I}, \text{ orientation}, & \exists\ \Gamma(\omega)\mathcal{I} = \Gamma(\mathbb{R}\omega) \end{array} \right\}$$

$\downarrow$

X

① $\Rightarrow$ ② Fix some "model" $Spin^c$-structure $\Gamma_*: V_* \to End(W_*)$. ← inner product space $V_*\cong V$

Fix some isom $E \cong P\times_{S^1} V_*$.

Obj $W := P\times_{S^1} W_*$

Γ induces a map $V \cong P\times_{S^1} V_* \to End(P\times_{S^1} W_*) = End(W)$.

§2 - $Spin^c$- & $Spin^c$-connections

§2.1 - Using Principal bundles

Recall: Starting with a connection on a principal G-bundle $P \to X$ "in" a $\mathbb{R}$-form $A \in \Omega^1(P, \mathfrak{g})$ satisfying some conditions.

↳ Let X be a Riemannian mfld, $\nabla^{LC}: C^\infty(X, TX) \to \Omega^1(X \otimes C^\infty(X, TX))$

$\rightsquigarrow \omega^{LC} \in \Omega^1(SO(n), so(n))$.

↳ $\exists$ $Spin(n) \to SO(n)$ is a double cover $\Rightarrow d\lambda_e : spin(n) \xrightarrow{\cong} so(n)$.

$\rightsquigarrow \tilde\omega^{LC} \in \Omega^1(P_{SO}, spin(n))$ "lifting" ω^{LC} ($(d\lambda_e)^{-1}\circ\omega^{LC}$)

$\rightsquigarrow$ **Verdict** $\exists!$ $Spin$-connection given a $Spin$-structure.

$P_{Spin} \to P_{SO} \to SO(n)$ (e.g. $Spin(n)$)

↳ **But** Don't work for $Spin^c$!

Because $\exists$: $Spin^c(n) \to SO(n)$ has kernel $S^1 := U(1)$

$\Rightarrow d\lambda_e : spin(n) \xrightarrow{\cong} so(n) \oplus i\mathbb{R}$

↳ To get a $Spin^c$-connection, we'll have to choose a "$U(1)$-connection."

Recall: $L_{\mathbb{C}} := P_{SO}\times_{S^1}\mathbb{C}$ ← determinant line bundle.

$= P_{U(n)}\times_{S^1}\mathbb{C}$

Def Fix a $Spin^c$-structure (W, Γ) on a Riemannian manifold X.

① A $Spin^c$-connection on (W, Γ) is a Hermitian connection on $L_{\mathbb{C}}$.

② A "$\quad$" is a Hermitian connection on W, s.t.

$$\nabla_v(\Gamma\psi)\mathcal{I} = (\Gamma\omega)(\nabla_v\mathcal{I}) + (\Gamma^{LC}_v\omega)\mathcal{I},$$

for $\mathcal{I}\in C^\infty(X, W)$, $v, \omega \in C^\infty(X, TX)$.

{Hermitian connection on $L_{\mathbb{C}}$}

↕

{Connection on principal $U(1)$-bundle}

$\exists!$ for $Spin$-connection are unique

{connection on principal $Spin^c$-bundle}

$\Leftarrow \exists$ by working in coordinates

①

Notation Write $\mathcal{A}(\Gamma)$ for the set of $Spin^c$-connections on (W, Γ).

↳ Salomon writes for $A\in\mathcal{A}(\Gamma)$, ① ∇A for the conn on $L_{\mathbb{C}}$

② ∇_A for the conn on W.

↳ The curvature of $\nabla A \in \mathcal{A}(\Gamma)$ is $F_A = 2^n\,tr(F_{\nabla_A}) \in \Omega^2(X, i\mathbb{R})$.

curvature of ∇_A, $F_{\nabla_A}\in\Omega^2(X, End(W))$

§3 - Dirac Operator

depends on $A\in\mathcal{A}(\Gamma)$

The Dirac operator is $D_A: C^\infty(X, W) \to C^\infty(X, W)$ given by

$$D_A\mathcal{I} = \sum_i (\Gamma e_i)(\nabla_{A, e_i}\mathcal{I}) \qquad \text{for a local o.n. frame } \{e_i\}$$

orthonormal

§3.1 - Self-adjointness

Recall: $L^2(X) \to \Omega^*(X) = C^\infty(X, \mathbb{R})$ given by $\to d, d^*$ adjoint to d.

$\rightsquigarrow \langle \Gamma(\cdot)\mathcal{I}, \mathcal{I}\rangle \in \Omega^1(X)$.

Compute: $d^*\langle \Gamma(\cdot)\mathcal{I}, \mathcal{I}\rangle = \langle \mathcal{I}, D_A\mathcal{I}\rangle - \langle D_A\mathcal{I}, \mathcal{I}\rangle$.

By Stokes Thm: $\int_X \langle D_A\mathcal{I}, \mathcal{I}\rangle$ real $= \int_X \langle \mathcal{I}, D_A\mathcal{I}\rangle$ real

↑ locally self-adjoint

§3.2 - Weitzenböck formula

Let $\rho: \Lambda^*T^*X\otimes\mathbb{C} \to End(W)$ by

$$\rho\Big(\sum_{i<j} r_{ij}\, e^i\wedge e^j\Big) = \sum_{i<j} r_{ij}\,\Gamma(e_i)\Gamma(e_j).$$

Then:

$$D_A\circ D_A = \nabla_A^*\nabla_A + \frac{1}{4}R + \rho(F_A)$$

Laplacian; Riemann scalar curvature

§3.3 - Fredholm

Recall: An bounded linear operator is Fredholm if both its kernel & cokernel are finite dim'l.

The Fredholm index is $\dim\ker - \dim\operatorname{coker}$.

$D_A: W^{1,2}(X, W) \to L^2(X, W)$ is Fredholm, index 0.

§3.4 - Splitting & consequences

Return to V being an oriented inner product space w/ $Spin^c$-structure (W, Γ).

For $\dim V = 2n$.

Then $\Gamma = e_{2n+1}\cdots e_1 \in C(V)$ is independent of choice of basis, $\Gamma^2 = (+1)^n$.

$\rightsquigarrow \Gamma(\Gamma) \in End(W)$ has eigenvalues $\pm i^n$.

$\rightsquigarrow W = W^+\oplus W^-$, where $W^\pm$ is the $\pm i^n$-eigenspace of $\Gamma\Gamma$.

Rmk This splitting is fully determined by $\{\mathcal{I}(1) \mid \mathcal{I}\in W^\pm\} \in W^\pm \quad \forall \cdot\in V$.

If (W, Γ) is a $Spin^c$-structure on X, then have $W = W^+\oplus W^-$.

Prop For $A\in\mathcal{A}(\Gamma)$, ∇_A preserves $W^\pm$.

Pf sketch Fix $\mathcal{I}\in C^\infty(X, W^\pm)$, $\beta: \mathbb{R}\to X$ a path. ETC $\nabla_{\dot\beta}\mathcal{I}\in C^\infty(X, W)$

Claim on frame $e_1,\dots,e_{2n}$ along β, only need to

$$\begin{aligned} \Gamma(e_{2n})\cdots\Gamma(e_1)\nabla_{\dot\beta}\mathcal{I} &= \nabla_{\dot\beta}\big(\Gamma(e_{2n})\cdots\Gamma(e_1)\mathcal{I}\big) \\ &= \nabla_{\dot\beta}\big(\Gamma(e_{2n}\cdots e_1)\mathcal{I}\big) \\ &= \nabla_{\dot\beta}\big(\Gamma(i^n)\mathcal{I}\big) \\ &= i^n\nabla_{\dot\beta}\mathcal{I} \quad \in C^\infty(X, W^\pm). \end{aligned}$$

$\rightsquigarrow D_A: C^\infty(X, W^+) \to C^\infty(X, W^-)$ — restrictions of D_A

Note: $D_A^*: C^\infty(X, W^-) \to C^\infty(X, W^+)$

$D_A\big|_{C^\infty(X, W^+)}$ ↔ $D_A^*\big|_{C^\infty(X, W^-)}$ adjoints of D_A.

$$\left(\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right) \mapsto \begin{pmatrix} \cdot & B \\ A & \cdot \end{pmatrix}$$

$\rho^+: \Lambda^{2,+}T^*X\otimes\mathbb{C} \to End(W^+)$

§4 - 4-dim'l Seiberg-Witten Eq'n

Let X be a 4-dim'l, oriented, Riemannian mfld w/ $Spin^c$-str. (W, Γ).

The SW eqn for $A\in\mathcal{A}(\Gamma)$, $\mathcal{I}\in C^\infty(X, W^+)$,

$$D_A\mathcal{I} = 0 \quad \text{and} \quad \rho^+\big(F_{\nabla_A}\big) = \langle \mathcal{I}, \mathcal{I}\rangle\mathcal{I} - \tfrac{1}{2}|\mathcal{I}|^2(1)$$

trace-free part of $\mathcal{I}\mathcal{I}^*$

trace-free part of ρ^+ (to be trace-free)