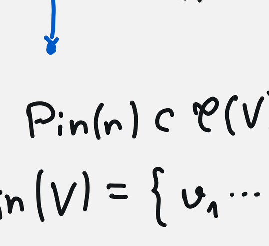
$R_v : V \longrightarrow V$
$\quad v \mapsto (u \mapsto -vuv^{-1})$

$R_v(u) = u - 2\frac{\langle u, v\rangle}{|v|^2} v \in V \longrightarrow$ the reflection along the hyperplane orthogonal to V



We might as well assume $|v| = 1$.

Def: $\mathrm{Pin}(n) \subset \mathcal{C}(V)^\times$ the subgroup generated by all $v \in V$, $|v| = 1$.

$\mathrm{Pin}(V) = \{ v_1 \cdots v_k \in \mathcal{C}(V) : v_i \in V, |v_i| = 1 \}$.

As above, $\mathrm{Pin}(V) \xrightarrow{ad} O(V)$. $\quad x \mapsto (u \mapsto \alpha(x) u x^{-1})$

- $ad(x)$ is orientation-preserving $\Leftrightarrow$ x is even

$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathcal{C}_{even}(V) = \{ v_1 \cdots v_{2k} \in \mathcal{C}(V) : v_i \in V, |v_i| = 1 \}$

Now we have a map $\mathrm{Spin}(n) \longrightarrow SO(n)$.

- **Prop:** We have an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{ad} SO(n) \longrightarrow 1.$$

$x \mapsto (u \mapsto xux^{-1})$

Moreover, $\mathrm{Spin}(n)$ is connected.

$\Rightarrow$ $\mathrm{Spin}(n)$ is the universal cover of $SO(n)$, $\pi_1(\mathrm{Spin}(n)) = 0$.

- surj: $O(n)$ is generated by reflections
- kernel: $x \in \mathrm{Spin}(n)$ s.t. $ad(x) = id$. $\quad xu = ux$
  x commutes with any $v \in V \Rightarrow x$ is scalar $\Rightarrow x = \pm 1$
- connectedness: $SO(n)$ is connected, it is enough to connect 1 to -1
  $\gamma(t) = \cos t + \sin t \, e_1 e_2 = \left(\cos\frac{t}{2} e_1 + \sin\frac{t}{2} e_2\right)\left(-\cos\frac{t}{2} e_1 + \sin\right)$
  $0 \le t \le \pi$

$\dim \mathrm{Spin}(n) = \dim SO(n) = \frac{n(n-1)}{2}$

Lie algebra: The matrix exponential $\exp: M_n(\mathbb{R}) \to GL(n)$

this generalises to any algebra. $\quad \exp: \mathcal{C}(V) \longrightarrow \mathcal{C}(V)^\times$
$\qquad x \mapsto \sum \frac{x^n}{n!}$

$\mathrm{Spin}(n) \subset \mathcal{C}(V)^\times$

Prop: $\mathfrak{spin}(n) = \mathrm{span}\{ e_i e_j : i \ne j \}$

$\mathrm{Spin}(V) \longrightarrow SO(V)$ $\qquad \xleftrightarrow{\text{diff.}}$ $\qquad \mathfrak{spin}(V) \xrightarrow{\cong} \mathfrak{so}(V)$
$x \mapsto (v \mapsto xvx^{-1})$ $\qquad\qquad\qquad y \mapsto (v \mapsto yv - vy)$

$x = e^{ty}$ $\qquad \frac{d}{dt}\Big|_{t=0}$

Remark: $\mathrm{Spin}(n)$ has no non-trivial characters. $\qquad \mathrm{Spin}(n) \to S^1$, $\mathbb{R}$

Complex Clifford algebras

$\mathcal{C}^c(V) := \mathcal{C}(V) \otimes_{\mathbb{R}} \mathbb{C}$, complexification of $\mathcal{C}(V)$

$\mathrm{Spin}(n) \subset \mathcal{C}(V) \subset \mathcal{C}^c(V) \qquad S^1 \subset \mathbb{C} \subset \mathcal{C}^c(V)$

Def: Let $\mathrm{Spin}^c(V) \subset \mathcal{C}^c(V)$ be generated by $\mathrm{Spin}(n)$ and S^1.

$\mathrm{Spin}^c(V) = \{ e^{i\theta} x : x \in \mathrm{Spin}(n), \theta \in \mathbb{R} \}$

$\mathrm{Spin}(n) \cap S^1 = \{\pm 1\} \Longrightarrow \mathrm{Spin}^c(n) = \frac{\mathrm{Spin}(n) \times S^1}{(1,1) \sim (-1,-1)} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1$

We still get a map $\mathrm{Spin}^c(n) \longrightarrow SO(n)$ $\qquad e^{i\theta}x = (-e^{i\theta})(-x)$
$\qquad e^{i\theta}x \mapsto ad(x)$

$\dim \mathrm{Spin}^c(n) = \dim \mathrm{Spin}(n) + 1$

Prop: We have exact sequences

$$1 \to S^1 \to \mathrm{Spin}^c(V) \to SO(V) \to 1$$
$$1 \to \mathrm{Spin}(V) \to \mathrm{Spin}^c(V) \xrightarrow{\delta} S^1 \to 1$$
$$\delta(e^{i\theta}x) = e^{2i\theta}$$

$\delta_*: \pi_1(\mathrm{Spin}^c(V)) \xrightarrow{\cong} \pi_1(S^1)$.

Lie algebra: $\mathfrak{spin}^c(n) = \mathfrak{spin}(n) \oplus i\mathbb{R} \subset \mathcal{C}^c(V)$

Classification of Clifford algebras

Prop: $\boxed{\mathcal{C}(\mathbb{R}^{n+8}) \cong \mathcal{C}(\mathbb{R}^n) \otimes M_{16}(\mathbb{R})}$ "Bott periodicity"
$\mathcal{C}^c(\mathbb{R}^{n+2}) \cong \mathcal{C}^c(\mathbb{R}^n) \otimes M_2(\mathbb{C})$

Corollary: $\mathcal{C}^c(\mathbb{R}^{2k}) = M_{2^k}(\mathbb{C})$, $\mathcal{C}^c(\mathbb{R}^{2k+1}) = M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$

- If $n = 2k$, $\mathcal{C}^c(\mathbb{R}^{2k})$ has a unique irreducible representation
  $\Gamma: \mathcal{C}^c(\mathbb{R}^{2k}) \xrightarrow{\cong} \mathrm{End}(\mathbb{C}^{2^k})$ "spin representation"
- If $n = 2k+1$, we get two
  $\mathcal{C}^c(\mathbb{R}^{2k+1}) \xrightarrow{\cong} \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k})$, $\xrightarrow{\Gamma_0} \mathrm{End}(\mathbb{C}^{2^k})$, $\xrightarrow{\Gamma_1} \mathrm{End}(\mathbb{C}^{2^k})$

(there is a convention to nail down one)

No problem: they restrict to the same rep. of $\mathrm{Spin}^c(\mathbb{R}^{2k+1})$.

Def: $\Delta_n := \mathbb{C}^{2^{\lfloor n/2 \rfloor}}$ the space of Dirac spinors

$\mathrm{Spin}(n) \subset \mathrm{Spin}^c(n) \longrightarrow \mathrm{End}(\Delta_n)$.

Mini rep. theory: we can see Δ_n as a Hermitian vector space $\langle \psi_1, \psi_2 \rangle$ such that the representation is unitary

$\mathrm{Spin}(n) \subset \mathrm{Spin}^c(n) \longrightarrow U(\Delta_n)$

Moreover: $\mathrm{Spin}(n) \longrightarrow SU(\Delta_n)$ $\qquad \det \Gamma: \mathrm{Spin}(n) \to S^1$ trivial

Clifford multiplication: $\Gamma: V \longrightarrow \mathrm{End}(\Delta_n)$

- Spin^c-equivariant $\quad V \otimes \Delta_n \longrightarrow \Delta_n \quad g \cdot (v \cdot \psi) = (gvg^{-1}) \cdot g\psi$
- Γ is anti-hermitian: $\Gamma(v) + \Gamma(v)^* = 0$
  $\langle v \cdot \psi_1, \psi_2 \rangle + \langle \psi_1, v \cdot \psi_2 \rangle = 0$

$\Gamma: V \longrightarrow \mathfrak{u}(\Delta_n)$ (in the sense $\mathrm{Lie}(U(\Delta_n))$)

Spin structures

X: smooth manifold

$Q \to X$ principal $SO(n)$-bundle $\longleftrightarrow$ $V \to X$ oriented real vector bundle of rank n w/ an inner product

Example: (X, g) oriented Riemannian manifold

$Q = SO(X) \to X$ bundle of oriented orthonormal frames $\longleftrightarrow$ $V = TX \to X$ tangent bundle

Def: A spin structure on Q is a principal $\mathrm{Spin}(n)$-bundle $P \to X$ and a double covering $P \to Q$ (over X) which intertwines the actions:

$$\begin{array}{ccc} P \times \mathrm{Spin}(n) & \longrightarrow & P \\ \downarrow & & \downarrow \\ Q \times SO(n) & \longrightarrow & Q \end{array}$$

this data $\quad P \times_{ad} SO(n) \xrightarrow{\cong} Q$

Two spin structures are isomorphic if $\exists\, P \xrightarrow{\cong} P'$ (over $Q \to X$) $\mathrm{Spin}(n)$-equivariant map.

- Given a spin structure, the spinor bundle $S := P \times_\rho \Delta_n$. $\quad \underbrace{P \times_\rho \Delta_n}_{\frac{P \times \Delta_n}{(p,\psi) \sim (pg,\, g^{-1}\psi)}}$
  $\rho: \mathrm{Spin}(n) \to SU(\Delta_n)$ $\qquad$ The fibres $S_x \cong \Delta_n$

$x \in X$: a choice of $(e_1, \dots, e_n) \in Q_x$ determines a local identification $T_x X \cong \mathbb{R}^n$

a choice of $p \in P_x$ determines $T_x X \cong \mathbb{R}^n$, $S_x \cong \Delta_n$.

Clifford multiplication $\quad \Gamma: V \longrightarrow \mathrm{End}(S)$

Prop: A spin structure exists $\Leftrightarrow$ $w_2(V) \in H^2(X; \mathbb{Z}_2)$ vanishes $\quad \hookrightarrow$ Stiefel-Whitney

Every compact, oriented, smooth 3-manifold has a spin structure.

But not all 4-manifolds have spin structures.

Ex: $\mathbb{C}P^2$ is not spin $\qquad$ Fact 1: $c_1(X) \equiv w_2(X) \bmod 2$
$\qquad$ Fact 2: $c_1(\mathbb{C}P^2) = 3L$

Def: We say X Riemannian, oriented has a spin structure if $Q = SO(X) \to X$ has a spin structure.

Def: A spin^c-structure on Q is a principal $\mathrm{Spin}^c(n)$-bundle $P^c \to X$ together with an isomorphism $P^c \times_{ad} SO(n) \xrightarrow{\cong} Q$. $\quad ad: \mathrm{Spin}^c(n) \to SO(n)$

As before: spinor bundle $S = P^c \times_\rho \Delta_n$, $\quad \Gamma: TX \longrightarrow \mathrm{End}(S)$

In this case we also have the characteristic line bundle $L := P^c \times_\delta \mathbb{C}$
$\delta: \mathrm{Spin}^c(n) \to S^1$

Remark: A spin structure determines a spin^c structure.

$\iota: \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}^c(n)$ If P is a spin structure, $P^c = P \times_\iota \mathrm{Spin}^c(n)$.

In this case, the line bundle L is trivial. $\qquad L = P^c \times_\delta \mathbb{C}$

$$1 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\iota} \mathrm{Spin}^c(n) \xrightarrow{\delta} S^1 \longrightarrow 1$$

Prop: A spin^c structure comes from a spin structure iff L is trivial.

Theorem: A spin^c structure exists $\Longleftrightarrow$ $w_2(V) \in H^2(X; \mathbb{Z}_2)$ has an integral lift to $H^2(X; \mathbb{Z})$

- If $P \to Q$ (over X) is a spin^c structure, $c_1(L)$ is an integral lift of $w_2(V)$.
- If $c \in H^2(X; \mathbb{Z})$ is an integral lift of $w_2(V)$, there is a spin^c structure on Q with $c_1(L) = c$.

Remark: Bockstein sequence

$$\cdots \to H^2(X; \mathbb{Z}) \xrightarrow{\bmod 2} H^2(X; \mathbb{Z}_2) \xrightarrow{\beta} H^3(X; \mathbb{Z}) \to \cdots$$
$$\mathrm{im}(\bmod 2) = \ker \beta$$

So the above $\Leftrightarrow$ $W_3 = \beta w_2 \in H^3(X; \mathbb{Z})$ vanishes
$\qquad$ integral Stiefel-Whitney class

Rem: $\{\mathrm{spin}^c\text{-struct.}\} \longrightarrow H^2(X; \mathbb{Z})$ is not a bijection
$\qquad \longmapsto c_1(L)$

"$P \otimes E$" $\qquad \tilde{L} = L \otimes E^{\otimes 2}$