Spin$^{(-c)}$ groups and spin$^{(-c)}$ structures

The story begins with $SO(n) = \{\text{rotations on } \mathbb{R}^n\}$

$\pi_1(SO(n)) = \mathbb{Z}/_2$ for $n \geq 3$

$\Rightarrow$ the universal covering $\quad Spin(n) \longrightarrow SO(n)$

$\hookrightarrow$ will construct this using Clifford algebras

V: finite-dim real v.s., oriented, inner product $\langle \cdot, \cdot \rangle$

Def: The Clifford algebra of V is: $\quad \mathcal{C}(V)$: (non-commutative) $\mathbb{R}$-algebra

$j: V \longrightarrow \mathcal{C}(V)$ linear map, $\quad j(v)^2 = -|v|^2$,

and it is universal among such pairs (A, u).

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "u"'] & \mathcal{C}(V) \arrow[d, dashed, "\exists !"] \\ & A \end{tikzcd}$$

Construction: $\quad \mathcal{C}(V) = \dfrac{TV}{\langle v \otimes v + |v|^2 \rangle}$

Presentation: $(e_1, \dots, e_n)$ orthonormal basis, $\quad e_i^2 = -1$, $\quad (e_i + e_j)^2 = -2 \Rightarrow e_i e_j = -e_j e_i$

$$\mathcal{C}(V) = \langle e_1, \dots, e_n \mid e_i^2 = -1,\ e_i e_j = -e_j e_i \rangle$$

In general: $v, w \in V$, $\quad vw + wv = -2\langle v, w \rangle$

Example: $V = \mathbb{R}^2$, $\quad \mathcal{C}(\mathbb{R}^2) = \langle e_1, e_2 \mid e_1^2 = e_2^2 = -1,\ e_1 e_2 = -e_2 e_1 \rangle \cong \mathbb{H}$

$e_1 = i, \quad e_2 = j, \quad e_1 e_2 = k$

Prop: $\mathcal{C}(V)$ is finite-dimensional: a basis of $\mathcal{C}(V)$ is

$$\{ e_I : I \subseteq \{1, \dots, n\} \}, \qquad I = (i_1, \dots, i_k), \quad e_I = e_{i_1} \cdots e_{i_k}$$

$i_1 < \dots < i_k$

$\dim \mathcal{C}(V) = 2^n$

$\mathcal{C}(\mathbb{R}^2) = \text{span}\{1, e_1, e_2, e_1 e_2\}$

- $\mathcal{C}(V)$ is not $\mathbb{Z}$-graded: $e_i^2 = -1$. But have a $\mathbb{Z}_2$-grading, $\mathcal{C}(V) = \mathcal{C}_{even}(V) \oplus \mathcal{C}_{odd}(V)$.

Involution $\alpha: \mathcal{C}(V) \to \mathcal{C}(V)$ that determines this decomp.

$v \mapsto -v$

$v_1 v_2 \mapsto v_1 v_2$

$v_1 \cdots v_k \mapsto (-1)^k v_1 \cdots v_k$

- Centre: $\quad Z(\mathcal{C}(V)) = \begin{cases} \mathbb{R} & \text{if } n \text{ even} \\ \mathbb{R} \oplus \mathbb{R} \cdot e_1 \cdots e_n & \text{if } n \text{ odd} \end{cases}$

$\mathcal{C}(V)^\times$ invertible elements $\rightsquigarrow$ open subset of $\mathcal{C}(V)$ $\qquad (\mathbb{C}^\times \subset \mathbb{C})$

$v \in V \setminus \{0\}$, $\quad v^2 = -|v|^2 \Rightarrow v^{-1} = -\dfrac{v}{|v|^2}$

Let $v \in V \setminus \{0\}$, consider the linear map $\quad R_v : V \longrightarrow V$

$u \longmapsto -vuv^{-1}$

- It is not clear that R_v lands in V