# The Thom Conjecture / The "Minimal Genus" Problem

§1 - Problem Statement

§2 - Adjunction Formulae

§3 - Sketch a proof of an easy case

## §1 - Thom Conjecture

**Thom Conjecture** - The minimal genus of a smoothly embedded surface representing $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$ is $\frac{(d-1)(d-2)}{2}$, $\cong \mathbb{Z}$

& is represented by a smooth algebraic curve.

**Generalised Thom Conj.** A symplectic surface $\Sigma$ $[\omega|_\Sigma > 0]$ in a sym. 4-mfld is genus-minimising its homology class.

↳ Kronheimer-Mrowka $d > 0$

↳ K-M; Morgan-Szabó-Taubes $[\Sigma].[\Sigma] > 0$

↳ Ozsváth-Szabó.

## §2 - Adjunction Formulae/(In)equalities

**Prop:** $(X,\omega)$ cpt symp 4-mfld, $\Sigma \subset X$ a symp. surface, then

$$-\chi(\Sigma) = \langle c_1(L_{\Gamma_{can}}), \Sigma \rangle + \Sigma \cdot \Sigma$$
$$c_1(L_{\Gamma_{can}}) = c_1(TX|_\Sigma)$$

**Pf:** $TX|_\Sigma = N_X\Sigma \oplus T\Sigma$, so

$$\langle c_1(TX), \Sigma \rangle = \langle c_1(N_X\Sigma), \Sigma \rangle + \langle c_1(T\Sigma), \Sigma \rangle$$

$$\Rightarrow \ -\langle c_1(L_{\Gamma_{can}}), \Sigma \rangle = \Sigma \cdot \Sigma + \chi(\Sigma). \qquad \square$$

**Thm** $X$ oriented cpt 4-mfld, $b^+ \geq 2$, $\Sigma \subset X$ smoothly embedded, connected, $g(\Sigma) \geq 1$. Then for all basic classes $c \in H^2(X)$,

$[SW(X,\Gamma) \neq 0, \ c = c_1(L_\Gamma)]$

$$-\chi(\Sigma) \geq |\langle c, \Sigma \rangle| + \Sigma \cdot \Sigma$$

↳ Holds for not-necessarily connected $\Sigma$ w/ no spherical cpts

↳ Hold all $\Sigma$ if $[\Sigma] \in H_2(X)$ is not torsion

↳ Similar result for $b^+ = 1$.

**Comparison to topologically setting.**

Let $X = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, has elliptic fibration $T^2 \hookrightarrow X \twoheadrightarrow \mathbb{CP}^1$.

↳ Smoothly, fibre class $F \in H_2(X)$ is rep'd by $g=1$.
Not torsion, $F^2 = 0$, $\rightsquigarrow$ obstructs $g=0$.

↳ Topologically $F^2 = 0$ — FQ Sphere Emb. Theorem $\to$ $F$ is top. rep'd a sphere.

↳ Consider $\alpha_d = dH + (d-1)(E_1 + \dots + E_9)$, $d \gg 0$.

$H$: $\mathbb{CP}^1 \subset \mathbb{CP}^2$; $E_i$: $\mathbb{CP}^1 \subset \overline{\mathbb{CP}^2}$

$$K = -3H - (E_1 + \dots + E_9)$$
$$\alpha_d^2 = d^2 - 9(d-1)^2 = 9 + 16d - 8d^2$$
$$K \cdot \alpha_d = -3d + 9(d-1) = 6d - 9.$$
$$\rightsquigarrow \ -\chi(\Sigma_d) \geq 24d - 8d^2$$

↳ Lee-Wilczyński: if $\pi_1(X) = 1$, any primitive class in $H_2(X)$ is rep'd by a torus topologically.

**Proof of Gen. Thom** * Seen before that $SW(X, \Gamma_{can}) = 1$, $SW(X, \overline{\Gamma}_{can}) = (-1)^{\frac{\chi+\sigma}{4}}$.

$\Rightarrow c = c_1(L_{\overline{\Gamma}_{can}})$ is a basic class.

So if $\Sigma \subset X$ is smoothly emb and $\Sigma'$ is symp, then

$$-\chi(\Sigma) \geq \Sigma \cdot \Sigma + |\langle c, \Sigma \rangle| \geq \Sigma' \cdot \Sigma' + \langle c, \Sigma' \rangle = -\chi(\Sigma').$$

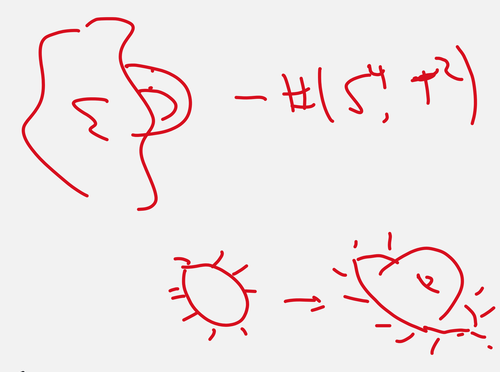
adj. ineq      adj. form.

## §3 - Sketch of an Easy Case

**Prop** If the adj. ineq. holds for zero self-intersection, then it holds for positive self-intersection.

**Pf** Fix $\Sigma \subset X$ w/ $\Sigma \cdot \Sigma = l > 0$, & $c$ a basic class.

WLOG, $\langle c, \Sigma \rangle \geq 0$, else replace $c \mapsto -c$.




Let $(X', \Sigma') := (X, \Sigma) \# l(\overline{\mathbb{CP}^2}, \overline{\mathbb{CP}^1})$, & compute that

$$c' := c + E_1 + \dots + E_l \in H_2(X')$$

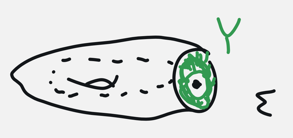
is a basic class. Then:

$$-\chi(\Sigma) = -\chi(\Sigma') \geq |\langle c', \Sigma' \rangle| = |\langle c, \Sigma \rangle + l| = |\langle c, \Sigma \rangle| + \Sigma \cdot \Sigma$$

**We can prove:** $X$ oriented 4mfld, $b^+ \geq 2$, $\Sigma \subset X$ smoothly emb, $g(\Sigma) \geq 1$, connected, $\Sigma \cdot \Sigma = 0$, then for all basic classes $c$, $-\chi(\Sigma) \geq |\langle c, \Sigma \rangle|$.

**Pf** ① Let $Y \subset X$ be a "cylinder around $\Sigma$".

↳ $Y \cong \Sigma \times S^1 \times I$




For $t \geq 1$, let $g_t$ be a metric on $X$ s.t.

$$g_t|_Y = h \oplus d\theta^2 \oplus dz^2$$

$h$: constant scalar curvature, vol = 1; $d\theta^2$: inf scal = 0, vol = 1; $dz^2$: flat, vol = t.

Since SW invariants are metric-indep, fix basic class $c$, & spin$^c$-str. $\Gamma$, $c = c_1(L_\Gamma)$, & find solutions $(A_t, \varphi_t)$ to SW-eqn. on $(X, \Gamma, g_t)$ w/ $\varphi_t \neq 0$.

② Gauss-Bonnet: $\text{scal}(h) = \int_\Sigma \text{scal}(h) = 4\pi\chi(\Sigma)$

$$\Rightarrow \text{scal}(g_t)|_Y = 4\pi\chi(\Sigma) + \text{scal}(d\theta^2).$$

$\sup_Y \left( \frac{1}{2} \max\{0, -\text{scal}\, g_t\} \right)$

$$\text{Shing: } \sup_Y |\varphi_t|^2 \leq \sup_Y \left( -\tfrac{1}{2}\, \text{scal}(g_t) \right) = -2\pi\chi(\Sigma). \qquad (*)$$

$$\text{By SW: } \sup_Y |F_t^+| \leq \sup_Y |\rho^+ F_t| \overset{SW}{=} \sup_Y \left( \langle \varphi_t, \cdot \rangle \varphi_t - \tfrac{1}{2}|\varphi_t|^2 \text{id} \right)$$

$\rho(e_i^* \wedge e_j^*) = \Gamma(e_i) \circ \Gamma(e_j)$

$$= \sup_Y \left( \tfrac{1}{2} |\varphi_t|^2 \right). \qquad (*)$$

$$\Rightarrow \sup_Y |F_t^+| \leq \overset{\sqrt{2}?}{-\pi}\chi(\Sigma).$$

$$\overset{\text{over } X}{\Rightarrow} \ \|F_t^+\|^2 \leq 2\pi^2\chi^2 t + S$$

← indep. of $t$, coming from $X \setminus Y$

$\|F_t\|^2 \leq \pi^2\chi^2 t$

③ By Chern-Weil theory: $c = \left[ \frac{i}{2\pi} F_t \right] \in H^2(X)$

Also, for $\eta \in \Omega^2(X; i\mathbb{R})$ $\|\eta\|^2 = \|i\eta\|^2 = \int_X (i\eta) \wedge *(i\eta) = -\int_X \eta \wedge *\eta$.

$$\Rightarrow \|F_t\|^2 = \|F_t^+\|^2 + \|F_t^-\|^2 = -\int_Y F_t^+ \wedge F_t^+ + \int_Y F_t^- \wedge F_t^-$$
$$= 2\|F_t^+\|^2 + \left( \int_X F_t^+ \wedge F_t^+ + \int_X F_t^- \wedge F_t^- \right)$$
$$= 2\|F_t^+\|^2 + \int_X F_t \wedge F_t$$
$$= 2\|F_t^+\|^2 - 4\pi^2 c^2.$$

$$\Rightarrow \|F_t\|^2 \leq \cancel{4}\pi^2\chi^2 t + 2S - 4\pi^2 c^2.$$

④ For closed real 2-form $\eta$, $\int_{\Sigma \times (s,t)} \eta$ is independent of $(s,t) \in S^1 \times I$.

$$\Rightarrow t \int_\Sigma \eta = \int_{S^1 \times I} \int_\Sigma \eta \leq \int_{\Sigma \times S^1 \times I} |\eta| \leq t^{\frac{1}{2}} \left( \int_Y |\eta|^2 \right)^{\frac{1}{2}} = t^{\frac{1}{2}} \|\eta\|.$$

$\Sigma \times S^1 \times I \cong Y$

$$\Rightarrow \|F_t\| \geq t^{\frac{1}{2}} \int_\Sigma iF_t = t^{\frac{1}{2}} \langle iF_t, \Sigma \rangle = 2\pi t^{\frac{1}{2}} \langle c, \Sigma \rangle.$$

⑤ $4\pi^2 t \langle c, \Sigma \rangle^2 \leq \|F_t\|^2 \leq \cancel{4}\pi^2\chi^2 t + S'$

$$\overset{t \to \infty}{\Rightarrow} \quad \langle c, \Sigma \rangle^2 \leq \chi^2 \quad \Rightarrow -\chi(\Sigma) \geq |\langle c, \Sigma \rangle|. \qquad \blacksquare$$