THE BBDG DECOMPOSITION THEOREM

STRATIFICATION OF ALG. MAPS

- algebraic map $f: X \to Y$ can be stratified; $\exists$ algebraic Whitney stratification $\mathcal{X}$ of X and $\mathcal{Y}$ of Y s.t. given any component S of $\mathcal{Y}$-stratum one has:
  - (a) $f^{-1}(S)$ is a union of connected components of strata of $\mathcal{X}$, each of which is mapped submersively to S by f; every fibre $f^{-1}(y)$ of f is stratified by its intersection w/ the strata of $\mathcal{X}$.
  - (b) For every point $y \in S$, $\exists$ Euclidean neighbourhood U of y in S and a stratum-preserving homeo $h: U \times f^{-1}(y) \to f^{-1}(U)$ s.t. $f|_{f^{-1}(U)} \circ h$ is the projection to U
- cpx alg. varieties and map can be compactified

DELIGNE'S DECOMPOSITION THM

Th^m 9.3.3 [Deligne]. Let $f: X \to Y$ be a smooth projective map of cpx alg. manifolds. Then

$$Rf_* \mathbb{Q}_X \simeq \bigoplus_{i \geq 0} \mathcal{H}^i(Rf_* \mathbb{Q}_X)[-i] \in D^b_c(Y)$$

and the local systems $R^i f_* \mathbb{Q}_X = \mathcal{H}^i(Rf_* \mathbb{Q}_X)$ are semisimple on Y.

↳ every fibre of is a nonsingular cpx proj. variety

Cor 9.3.5. Let $f: X \to Y$ be a smooth projective map of cpx alg. manifolds. Then,

$$H^*(X, \mathbb{Q}) \cong \bigoplus_{i \geq 0} H^{*-i}(Y; R^i f_* \mathbb{Q}_X)$$

$\Rightarrow$ Leray spectral sequence

$$E_2^{p,q} = H^p(Y; R^q f_* \mathbb{Q}_X) \Rightarrow H^{p+q}(X, \mathbb{Q}) \quad \text{degenerates on } E_2\text{-page}$$

SEMI-SMALL MAPS (s'pose $f: X \to Y$ proper surjective, X nonsingular and pure-dim)

- fix a stratification $Y = \sqcup_\lambda S_\lambda$ of Y, let $s_\lambda \in S_\lambda$ be any of its points
- $f^{-1}(S_\lambda) \to S_\lambda$ is topologically locally trivial fibration $\Rightarrow$ fibres of f over S_λ have constant dim

Def^n 9.3.7. The defect of semi-smallness $r(f)$ of f is defined by

$$r(f) = \max_\lambda \{2\dim_{\mathbb{C}} f^{-1}(s_\lambda) + \dim_{\mathbb{C}} S_\lambda - \dim_{\mathbb{C}} X\} \geq 0.$$

The map f is called semi-small if $r(f) = 0$ $\Leftrightarrow$ for every λ, $\dim_{\mathbb{C}} S_\lambda + 2\dim_{\mathbb{C}} f^{-1}(s_\lambda) \leq \dim_{\mathbb{C}} X$.

A stratum S_λ of Y is called relevant if $\dim_{\mathbb{C}} S_\lambda + 2\dim_{\mathbb{C}} f^{-1}(s_\lambda) = \dim_{\mathbb{C}} X$.

A semi-small map with no relevant strata of positive codimension is called small.

↳ surjective maps between cpx surfaces are always semi-small

! projective birational maps from a holomorphic symplectic nonsingular variety

Prop 9.3.13. Let $f: X \to Y$ be a small map, with X nonsingular of cpx dim n. Let $U \subseteq Y$ be the nonsingular dense open subset over which f is a covering map. Then

$$Rf_* \mathbb{Q}_X[n] \simeq IC_Y(\mathcal{L})$$

where $\mathcal{L}$ is the local system $(f_* \mathbb{Q}_X)|_U$.

Cor 9.3.14. If $f: X \to Y$ is a small resolution, then

$$Rf_* \mathbb{Q}_X[\dim_{\mathbb{C}} X] \simeq IC_Y .$$

In particular,

$$IH^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q})$$

↳ if they exist, small resolutions are not necessarily unique; if Y has several small resolutions, their cohomologies are isomorphic as group, but not always as rings (i.e. there is no natural product on intersection cohomology)

Prop 9.3.18. Let $f: X \to Y$ be a proper surjective map, with X nonsingular of pure cpx dim n. Then,

$$Rf_*(\mathbb{Q}_X[n]) \in {}^p\mathcal{D}^{\leq r(f)}(Y) \cap {}^p\mathcal{D}^{\geq r(f)}(Y),$$

i.e. ${}^p\mathcal{H}^i(Rf_*(\mathbb{Q}_X[n])) = 0$ for $i \neq [-r(f), r(f)]$.

$\Rightarrow$ if f semi-small, then $Rf_*(\mathbb{Q}_X[n]) \in \mathrm{Perv}(Y)$.

- let S be a relevant stratum of f, $s \in S$ and define

$$\mathcal{L}_S := (R^{n-\dim_{\mathbb{C}} S} f_* \mathbb{Q}_X)|_S \quad \Rightarrow \mathcal{L}_S \text{ local system on } S \text{ (f is proper)}$$

↳ $IC_{\bar{S}}(\mathcal{L}_S)$ is a semisimple perverse sheaf

Th$^{\underline{m}}$ 9.3.20. Let $f: X \to Y$ be a proper surjective semismall map w/ X nonsingular of pure dim n. Let $\mathcal{Y}$ be a Whitney stratification of Y w.r.t. which f is stratified, denote by $\mathcal{Y}_{rel} \subset \mathcal{Y}$ the set of relevant strata of f. There is a canonical isomorphism in $\mathrm{Perv}(Y)$,

$$Rf_* \mathbb{Q}_X[n] \simeq {}^p\mathcal{H}^0(Rf_* \mathbb{Q}_X[n]) \simeq \bigoplus_{S \in \mathcal{Y}_{rel}} IC_{\bar{S}}(\mathcal{L}_S)$$

<u>DECOMPOSITION THEOREM</u>

Th$^{\underline{m}}$ 9.3.25 [BBDG]. Let $f: X \to Y$ be a proper map of cpx alg. varieties, with X pure-dimensional. Then,

(i) there is a (non-canonical) isomorphism in $D^b_c(Y)$

$$Rf_* IC_X \simeq \bigoplus_i {}^p\mathcal{H}^i(Rf_* IC_X)[-i] ;$$

(ii) each ${}^p\mathcal{H}^i(Rf_* IC_X)$ is semisimple in $\mathrm{Perv}(Y)$, hence

$$ {}^p\mathcal{H}^i(f_* IC_X) \simeq \bigoplus_{S\in\mathcal{Y}} IC_{\bar{S}}(\mathcal{L}_{i,S})$$