Talk 8: Degree 0 DT invariants (of CY3)

Goal for today: Compute $DT_n^X := (-1)^n \chi(\mathrm{Hilb}^n(X)) := (-1)^n \underline{\underline{X^{[n]}}}$.

(for X local CY.)

<u>Defn</u>: (Trace) $\quad tr^i : \mathrm{Ext}^i(E,E) \longrightarrow H^i(X, \mathcal{O}_X)$

and $\underline{\underline{\mathrm{Ext}^i(E,E)_0}} = \ker(tr^i)$.

<u>Ex</u>: E Gieseker stable sheaf on X, then tr^1 can be identified w/ the differential of the determinant map $(\det : \mathcal{M} \longrightarrow \mathrm{Pic}(X))$.

Upshot: Ext^1_0 and Ext^2_0 encode the tangent and obstruction information about the deformation theory of E w/ fixed det.

Today we will mostly discussing CY3 (but why?

$\Rightarrow$ enum geo for X CY3 becomes "rich". The $d^{virt} = 0$.

$\Rightarrow$ This follows from Serre duality:

$$\mathrm{Ext}^1(E,E) \cong \underbrace{\mathrm{Ext}^2(E,E)}_{}{}^{\vee}$$

<u>Defn</u>: (CY3) A CY3 is a nonsingular, quasiproj variety X w/ $\dim_{\mathbb{C}} X = 3$ satisfying $\omega_X := \Lambda^3 \Omega_X^1 \cong \mathcal{O}_X$ and ($H^i(X, \mathcal{O}_X) = 0$ for $0 < i < 3$).

Ex: - $\mathbb{C}^3$ $\not$

- quintic in $\mathbb{P}^4$ (smooth)

- Tot (ω_S) $\not$

- general complete intersections.

✓

3) If X toric and CY $(\mathbb{C}^\times)^3 \subset X$ then the sym p.o.t above is $(\mathbb{C}^\times)^3$ - equivariant.

Thm: Let X smooth 3-fold and $n \in \mathbb{Z}_{\geq 0}$

1) If X projective and is either CY or satisfies $H^{>0}(X, \mathcal{O}_X) = 0$ (e.g. X is toric), then $X^{[n]}$ has 0-dim p.o.t.

2) In CY case sym. p.o.t.

__Defn:__ Quasiprojective CY3 which is toric then we call it a local CY3.

__Ex:__ - Tot (ω_S) S toric surface

- Tot $(L_1 \oplus L_2)$ over $\mathbb{P}^1$ s.t. $L_1 \otimes L_2 \cong \mathcal{O}_{\mathbb{P}^1}(-2)$

__Thm:__ X a smooth ~~projective~~ 3-fold, then

$$H^i(X, \mathcal{O}_X) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_X) = 0 \quad , i > 0$$

hence $\mathrm{tr}^i = 0$ for $i > 0$ and in particular

$$\mathrm{Ext}^i(E, E)_0 = \mathrm{Ext}^i(E, E) \quad , \quad i > 0$$

and (after some work) for $\dim_{\mathbb{C}} Z = 0$, $Z \subset X$ of length n
$\chi(\mathcal{O}_Z) = n$ we get $\mathrm{Hom}(\mathcal{I}, \mathcal{I})_0 = \mathrm{Ext}^3(\mathcal{I}, \mathcal{I})_0$.

Rmk: This theorem also holds for local CY3.

$$\mathrm{Ext}^3(\mathcal{I},\mathcal{I}) \neq \mathrm{Ext}^3(\mathcal{I},\mathcal{I})_0 \text{ but}$$

$$\mathrm{Hom}(\mathcal{I},\mathcal{I})^\vee = \mathrm{Ext}^3(\mathcal{I},\mathcal{I}) = \mathbb{C} \text{ and thus}$$

we know $\mathrm{Ext}^3_0 = 0$.

Thm: X smooth proj 3-fold s.t. $H^i(X,\mathcal{O}_X)=0$ for $i=1,2$

Then for every $n \geq 0$ we have:

$$\mathrm{Hilb}^n X \cong \mathcal{M}_X(1,0,0,-n)$$

w/ $\mathcal{M}$ moduli of Gieseker stable sheaves on X w/ $\mathrm{ch}\,\underline{E} = (1,0,0,-n)$

Proof: $[E] \in \mathcal{M}$ will always of the form $E = \mathcal{I} \otimes \mathcal{L}$, where

$\mathcal{I}$ will be ideal of colength n and $\mathcal{L}$ is a degree 0 line bundle.

$\Rightarrow$ $\underline{\mathcal{L} = \mathcal{O}_X}$ as $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$.

This means $H^1(X, \mathcal{O}_X) \Rightarrow \mathrm{Pic}^0(X)$ is trivial!

$\Rightarrow$ By our assumption on X, $\mathcal{M}$ is a moduli of ideal

sheaves.

$$[\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z]$$

Define a map $f : X^{[n]} \longrightarrow \mathcal{M}$ and show that f is étale.

$$[\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z] \mapsto \tilde{I}_Z$$

Apply formal criterion for étale maps and show this at $[\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z]$. $\square$

Rmk: The obs. theory of $X^{[n]}$ is bad. But $\mathcal{M}$ has good obs. theory

$$T^i|_{I_Z \in \mathcal{M}} = \mathrm{Ext}^i(I, I)_0 .$$

$\Longrightarrow$ Find p.o.t for $\mathcal{M}_X$ i.e. $X^{[n]}$

Rmk: We can get "for free" that there exists a p.o.t

$$\Rightarrow [X^{[n]}]^{virt} \in A_0(X^{[n]})$$

Defn: The degree 0 DT inv. of the smooth, proj. 3-fold X are defined as

$$DT_n^X := \int_{[X^{[n]}]^{virt}} 1 \in \mathbb{Z}.$$

Notation: $\mathbb{T} = (\mathbb{G}_m^g) = (\mathbb{C}^\times)^g$ w/ cocharacter lattice $\hat{\mathbb{T}} = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T}) \cong \mathbb{Z}^g$. Let $K_0^{\mathbb{T}}(pt)$ be the K-group of the cat. of fin. dim $\mathbb{T}$-reps. Let $t_1, \dots, t_g$ are the generators w/ t_j being the class of maps $\mathbb{T} \longrightarrow (\mathbb{C}^\times)$.
$$(\theta_1, \dots, \theta_g) \mapsto \theta_j$$

Set $s_j = c_1^T(t_j)$ then $H_T^* = \mathbb{Q}[s_1, \dots, s_g]$.

We will compute DT_n^X via torus localization on local CY.

$$[X^{[n]}]^{virt} = \iota_* \sum_{I_Z} \frac{[S(I_Z)]^{vir}}{e^T(N_Z^{vir})} \in A_0^T(X^{[n]}) \otimes H_T$$

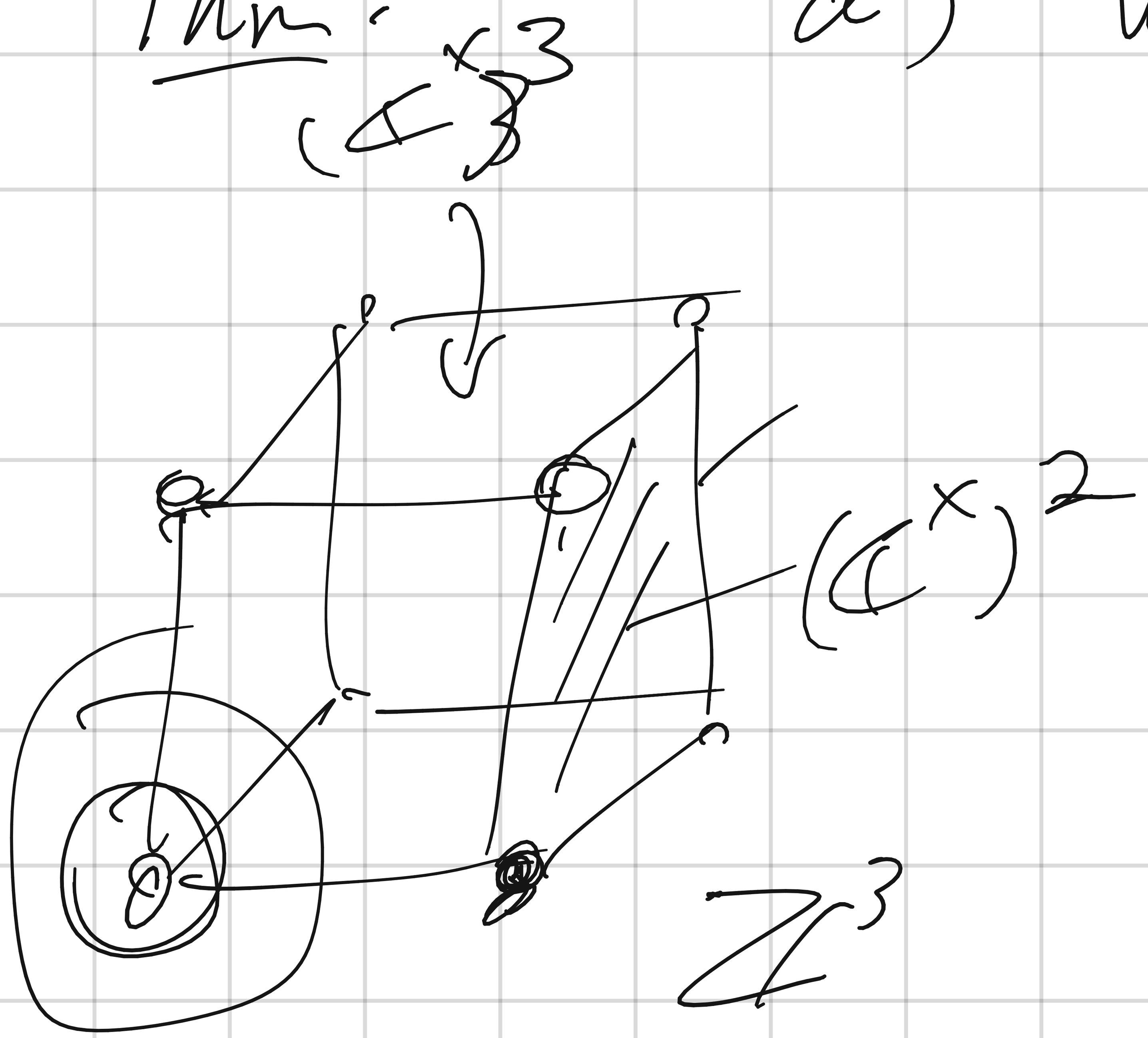
where: 1) $\iota : X^{[n]^T} \hookrightarrow X^{[n]}$

2) The sum is over T-fixed ideals

3) $S(I_Z)$ connected closed subset of $(X^{[n]})^T$ at $I_Z \in M_X^T$.

4) $N_Z^{vir} = \mathbb{E}^\vee|_{I_Z}^{mov} \in K_0^T(pt)$

Thm: a) We have the iso



$$X^{[n]\pi} \cong \bigsqcup_{\sum_\alpha n_\alpha = n} \prod_{\alpha \in \Delta(X)} U_\alpha^{[n]\pi}$$

$U_\alpha \cong \mathbb{C}^3 \supset (\mathbb{C}^\times)^3$

b) At $I_Z \in M^\pi$ Ext^1, Ext^2 vanish.

c) Same holds for $\underline{\pi_0} \subset \pi$ preserving the volume form on X.

Our goal $DT_X(q) = \sum_{n\geq 0} DT_n^X q^n \in \overbrace{\mathbb{Q}(s_1,s_2,s_3)[[q]]}^{\mathcal{H}_T}$

$s_i = c_1^T(t_i)$

If X is loc CY then $DT_X(q) \in \mathbb{Z}[[q]]$

Thm: $DT_X(q) = \left(\prod_{m\geq 1}\left(1-(-q)^m\right)^m\right)^{\chi(X)} = \left(M(-q)\right)^{\chi(X)}$

$M(q_0)^{\sim} \cdot M(q_1) \cdot M$

$\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3 = 0$

If X is proj but not toric then

$$DT_X(q) = M(-q)^{\int_X c_3(T_X \otimes \omega_X)}$$

In fact: ${}_{\mathrm{loc}}DT^{T}_{\mathbb{C}^3}(q) = M(-q)^{-\frac{(\tilde{s}_1+\tilde{s}_2)(\tilde{s}_2+\tilde{s}_3)(\tilde{s}_1+\tilde{s}_3)}{\tilde{s}_1\tilde{s}_2\tilde{s}_3}}$ Szendrői-conj.