3.2 Tor and Ext

Recall: F projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and M a G-module. Then

$$H_*(G,M) := H_*(F \otimes_G M)$$

$$H^*(G,M) := H^*(\mathrm{Hom}_G(F,M))$$

We generalise $H_*(G,-)$ and $H^*(G,-)$ by introducing $\mathrm{Tor}^G_*(-,-)$ and $\mathrm{Ext}^*_G(-,-)$.

Derived functors

Consider a functor $F: \mathcal{A} \longrightarrow \mathcal{B}$.

Def: the LEFT DERIVED FUNCTORS L_iF are defined as follows:

given $A \in \mathrm{Ob}(\mathcal{A})$ take a projective res. $P_\bullet \to A$, then $L_iF(A) := H_i(F(P_\bullet))$. Given a morphism $\alpha: A \to B$ define $L_iF(\alpha): L_iF(A) \longrightarrow L_iF(B)$ as $L_iF(\alpha) := H_i(F(\alpha_\bullet))$.

The RIGHT DERIVED FUNCTORS R^iF are defined as:

take an injective res. $A \to Q_\bullet$, then $R^iF(A) := H^i(F(Q_\bullet))$ and $R^iF(\alpha) = H^i(F(\alpha_\bullet))$ for α morphism.

Side note: an R-mod. Q is INJECTIVE iff the functor $\mathrm{Hom}(-,Q)$ is exact:

$$\begin{tikzcd} A \arrow[r, "\lambda_A"] \arrow[d, hook] & Q \\ B \arrow[ur, "\lambda_B"'] & \end{tikzcd}$$

A an R-mod.

Def: define $\mathrm{Tor}_i^R(A,-)$ as L_iF with F the right exact functor $A \otimes_R -$.

$$\mathrm{Tor}_i^R(A,B) := L_i(A \otimes_R -)(B) = H_i(A \otimes_R P_\bullet) \quad \text{with } P_\bullet \to B \text{ proj. res}$$

Def: define $\mathrm{Ext}^i_R(A,-)$ as R^i of the left exact functor $\mathrm{Hom}(A,-)$.

$$\mathrm{Ext}^i_R(A,B) := R^i(\mathrm{Hom}(A,-))(B) = H^i(\mathrm{Hom}(A,Q_\bullet)) \quad \text{with } B \to Q_\bullet \text{ inj. res.}$$

↳ Ext can be defined also as R^i of the controvariant functor $\mathrm{Hom}(-,B)$:

$$\mathrm{Ext}^i_R(A,B) = R^i(\mathrm{Hom}(-,B))(A) = H^i(\mathrm{Hom}(P_\bullet, B)) \quad \text{with } P_\bullet \to A \text{ proj. res.}$$

We can apply the following proposition on derived functors to Tor and Ext.

Prop: let F be right exact, then $L_0F \cong F$ and for a SES $0 \to A \to B \to C \to 0$ there is an associated LES $\dots \to L_iF(A) \to L_iF(B) \to L_iF(C) \to L_{i-1}(A) \to \dots$ ending with $F(C) \to 0$.

Similarly, if F is left exact then $R^0F \cong F$ and a SES induces a LES of the form

$$0 \to F(A) \to \dots \to R^iF(A) \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to \dots$$

Hence, we have:

1. $\mathrm{Tor}_0^R(A,B) = A \otimes_R B$ and for SES $0 \to B_1 \to B_2 \to B_3 \to 0$ we get

$$\dots \to \mathrm{Tor}_i^R(A,B_1) \to \mathrm{Tor}_i^R(A,B_2) \to \mathrm{Tor}_i^R(A,B_3) \to \dots \to A\otimes_R B_1 \to A\otimes_R B_2 \to A \otimes_R B_3 \to 0.$$

2. $\mathrm{Ext}_R^0(A,B) = \mathrm{Hom}_R(A,B)$ and for SES $0 \to B_1 \to B_2 \to B_3 \to 0$ we get

$$0 \to \mathrm{Hom}(A,B_1) \to \mathrm{Hom}(A,B_2) \to \mathrm{Hom}(A,B_3) \to \mathrm{Ext}_R^1(A,B_1) \to \dots$$

or

$$0 \to \mathrm{Hom}(A_3,B) \to \mathrm{Hom}(A_2,B) \to \mathrm{Hom}(A_1,B) \to \mathrm{Ext}_R^1(A_3,B) \to \dots$$

Cohomology via Ext and Tor

Consider the category of G-modules, then Tor_i^G and Ext_G^i generalise homology and cohomology. Consider M, N two G-modules with proj. res. $F_\bullet \to M$ and $P_\bullet \to N$, then

$$\mathrm{Tor}_i^G(M,N) = H_i(F \otimes_G N) = H_i(F \otimes_G P) = F_i(M \otimes_G P)$$

and

$$\mathrm{Ext}_G^i(M,N) = H^i(\mathrm{Hom}_G(F,N))$$

We recover homology and cohomology by taking $M = \mathbb{Z}$:

$$\mathrm{Tor}_i^G(\mathbb{Z},N) = H_i(F\otimes_G N) = H_i(G,N)$$

$$\mathrm{Ext}_G^i(\mathbb{Z},N) = H^i(\mathrm{Hom}_G(F,N)) = H^i(G,N)$$

So we could have defined the cohomology of G with coeff. in M by:

$$H^i(G,M) = \mathrm{Ext}_G^i(\mathbb{Z},M).$$

ex: $H^0(G,M) = \mathrm{Ext}_G^0(\mathbb{Z},M) = \mathrm{Hom}_G(\mathbb{Z},M) = M^G$

$H_0(G,M) = \mathrm{Tor}_0^G(\mathbb{Z},M) = \mathbb{Z}\otimes_G M = M_G.$

Prop: Let M, N be G-modules. If M is $\mathbb{Z}$-torsion-free then

$$\mathrm{Tor}_i^G(M,N) \approx H_i(G, M\otimes N),$$

with G acting diagonally on $M \otimes N$.

If M is $\mathbb{Z}$-free, then $\mathrm{Ext}_G^i(M,N) \approx H^*(G, \mathrm{Hom}(M,N))$, G acts diagonally on $\mathrm{Hom}(M,N)$.