Functoriality

Def.: A category $\mathcal{C}$ consists of a class of objects $Ob(\mathcal{C})$ and a class of morphisms $Mor(\mathcal{C})$ together with assignments

think a set that is smaller than a given cardinal, Grothendieck universes

ⓐ $d_1 : Mor(\mathcal{C}) \longrightarrow Ob(\mathcal{C})$ a source map

ⓑ $d_0 : Mor(\mathcal{C}) \longrightarrow Ob(\mathcal{C})$ a target map

ⓒ $s_0 : Ob(\mathcal{C}) \longrightarrow Mor(\mathcal{C})$ the identity map s.t. $d_0 s_0(x) = d_1 s_0(x) = x \quad \forall x \in Ob(\mathcal{C})$

ⓓ $\circ : Mor(\mathcal{C}) \times_{Ob(\mathcal{C})} Mor(\mathcal{C}) \longrightarrow Mor(\mathcal{C})$ called composition and written as $(f,g) \longmapsto g \circ f$ where $Mor(\mathcal{C}) \times_{Ob(\mathcal{C})} Mor(\mathcal{C}) = \{(f,g) \in Mor(\mathcal{C}) \times Mor(\mathcal{C}) \mid d_0 f = d_1 g\}$

such that

$$d_0(f) = d_1(g) \;\&\; d_0(g) = d_1(h) \Rightarrow (h \circ g) \circ f = h \circ (g \circ f)$$
$$f \circ s_0(d_1(f)) = f = s_0(d_0(f)) \circ f$$

Let us write $X_0 = Ob(\mathcal{C})$ and $X_1 = Mor(\mathcal{C})$ we get a diagram

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \quad \text{with} \quad d_0 s_0 = 1, \quad d_1 s_0 = 1 .$$

A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ consists of maps $X_0 \to Y_0$ and $X_1 \to Y_1$ making the following diagrams commute

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[d] \\ Y_0 & Y_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \end{tikzcd} \qquad \begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "\circ"] \arrow[d] & X_1 \arrow[d] \\ Y_1 \times_{Y_0} Y_1 \arrow[r, "\circ"] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r] \arrow[d] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd}$$

three diagrams in one! all commute!

A natural transformation between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment $X_0 \xrightarrow{\eta} Y_1$ such that

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift right, "G"'] \arrow[d, shift left, "F"] & X_0 \arrow[r, "\eta"] & Y_1 \arrow[d, "d_1"] \\ Y_1 \arrow[rr, shift left, "d_0"] \arrow[rr, shift right, "d_1"'] & & Y_0 \end{tikzcd}$$

"commutes" two diagrams in one, all commute

$$\begin{tikzcd} X_0 \arrow[r, "(\eta \circ d_0) \times G"] \arrow[d, "F \times (\eta \circ d_1)"'] & Y_1 \times_{Y_0} Y_1 \arrow[d, "\circ"] \\ Y_1 \times_{Y_0} Y_1 \arrow[r, "\circ"] & Y_1 \end{tikzcd}$$

commutes

$C_*: Grp \to Ch(Ab)$ functorial

$M \in Ab$

$f: G \to G'$

$G \to End(M)$

$\downarrow$

G'

makes $\mathbb{Z}G'$ into a $\mathbb{Z}G$-module $\rightsquigarrow$ $f: \mathbb{Z}G \to \mathbb{Z}G'$

$f^*: {}_{\mathbb{Z}G'}Mod \to {}_{\mathbb{Z}G}Mod$

$P_\bullet \to \mathbb{Z}$ a projective resolution over $\mathbb{Z}G'$

$$\begin{tikzcd} {}_{\mathbb{Z}G'}Mod \arrow[r, "f^*"] & {}_{\mathbb{Z}G}Mod \arrow[l, bend right, "- \otimes_{\mathbb{Z}G} \mathbb{Z}G'"'] \arrow[l, bend left, "Hom_{\mathbb{Z}G'}(\mathbb{Z}G, -)"] \end{tikzcd}$$

extension of scalars

$\perp$

restriction of scalars

$\perp$

coextension of scalars

$F_\bullet \to \mathbb{Z}$ $\quad$ $F'_\bullet \to \mathbb{Z}$

$G \xrightarrow{\alpha} G'$

take $\alpha^*(F'_\bullet) \to \alpha^*(\mathbb{Z})$ because α^* is both left and right adjoint it preserves both kernels and cokernels $\alpha^*(\mathbb{Z}) \cong \mathbb{Z}$ as $\mathbb{Z}G$-modules

we get a chain map $F_\bullet \xrightarrow{\tau} \alpha^*(F)$ by our construction

$$\begin{tikzcd} F_\bullet \arrow[r, "\tau"] \arrow[dr] & \alpha^*(F) \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

from the uniqueness proof

well-defined up to homotopy

$$\begin{tikzcd} {}_{\mathbb{Z}G'}Mod \arrow[r, "\alpha^*"] \arrow[dr, "- \otimes_{\mathbb{Z}G'} \mathbb{Z}"'] & {}_{\mathbb{Z}G}Mod \arrow[d, "- \otimes_{\mathbb{Z}G} \mathbb{Z}"] \\ & Ab \end{tikzcd}$$

?

$\tau(g.x) = \alpha(g)\,\tau(x)$

$\Rightarrow$ we get a map $F_G \to F'_G$ unique up to homotopy

$\Rightarrow$ we get a unique map $H_*(G) \to H_*(G')$

This is functorial by uniqueness

Proposition: Fix $g_0 \in G$ and let $\alpha : \begin{array}{l} G \longrightarrow G \\ g \longmapsto g_0 g g_0^{-1} \end{array}$. Then $\alpha_* : H_*(G) \to H_*(G)$ is the identity.

Proof: Let $F_\bullet \to \mathbb{Z}$ be a projective resolution

$\alpha : S \to R$ a ring homomorphism   Let $M \in {}_R\mathrm{Mod}$ and $N \in {}_S\mathrm{Mod}$.

we get the following structure R is an (R,S)-bimodule:

$r.s = r\alpha(s) \rightsquigarrow r(ss') = r\alpha(s)\alpha(s') = (r.s).s'$

We obtain the tensor

$$R \otimes_S - : {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$$

R also becomes a left S-module via $s.r = \alpha(s)r$. (I)

This means ${}_R\mathrm{Mod}(R,M)$ becomes a left S-module via $s.f : r \mapsto f(\alpha(s)r)$. (II)

As abelian groups $M \cong {}_R\mathrm{Mod}(R,M)$ and in fact ${}_R\mathrm{Mod}(R,M) \cong \alpha^*(M)$.

$$\begin{tikzcd} {}_R\mathrm{Mod} \arrow[r, "\alpha^*"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & {}_S\mathrm{Mod} \arrow[l, bend right=50, "\alpha_! \cong R\otimes_S -"'] \arrow[l, bend left=50, "\alpha_* \cong {}_S\mathrm{Mod}(R,-)"] \end{tikzcd}$$

hom - tensor adjunction

so infact the adjunction (restriction of scalars) $\dashv$ (extension of scalars) is an instance of the hom $\dashv$ tensor adjunction

Now we also have an R-module structure on ${}_S\mathrm{Mod}(R,N)$ via

$r'.f : r \mapsto f(rr')$ (III)   note $((r''r').f)(r) = f(rr''r') = (r''.(r'.f))(r) =$

$= (r'.f)(rr'') = f(rr''r')$

We wish to show that ${}_S\mathrm{Mod}(R,-) \dashv \alpha^*$:

$${}_R\mathrm{Mod}(M, {}_S\mathrm{Mod}(R,N)) \cong {}_S\mathrm{Mod}(\alpha^*(M), N)$$

$$f \longmapsto (m \overset{\varphi}{\mapsto} f(m)(1))$$

(a) $f(r._R m) = r._R f(m)$

&

(b) $f(m)(s.r) = s.(f(m)(r))$

$\varphi(s._S m) = f(s._S m)(1) \overset{(II)}{=} f(\alpha(s)._R m)(1)$

$\overset{(a)}{=} (\alpha(s)._R f(m))(1) \overset{(III)}{=} f(m)(1\alpha(s)) =$

$\overset{(I)}{=} f(m)(s.1) \underset{(b)}{=} s.(f(m)(1))$

$$g \longleftarrow \phi : \alpha^*(M) \to N$$

$g(m) : r \mapsto \phi(r.m)$ $\phi(s._S m) = s.\phi(m)$

 $\| \; \phi(\alpha(s)._R m)$

$g(m)(s.r) = g(m)(\alpha(s)r)$

$= \phi((\alpha(s)r).m) =$

$= \phi(\alpha(s)._R(r._R m)) = \phi(s._S(r._R m))$

$= s.\phi(r._R m) = s.g(m)(r)$

$(r_0.g(m))(r) = g(m)(rr_0) = \phi((rr_0).m) = \phi(r.(r_0.m)) = g(r_0 m)(r)$

so g satisfies both (a)&(b) which is what we needed