Spectral Networks & Abelianisation

AGQ Conference 2025

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0) The Moduli Space of Flat Connections

Fix genus g Riemann surface C with $n \geq 1$ punctures, and a connected, reductive group G.

dfn. The moduli space of flat connections on C is

$$\mathcal{M}_C(G) := \{\text{flat } G\text{-connections on } C\} \Big/ \begin{matrix}\text{gauge}\\ \text{trafo.}\end{matrix}$$

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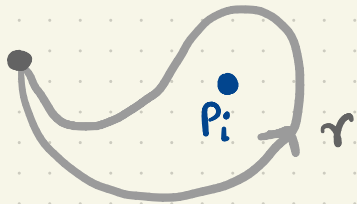
Recall: For a (flat) connection $\nabla = d + A$ and a curve $\gamma \subset C$,

dfn. $\mathrm{hol}_\gamma \nabla = P\exp \oint_\gamma A \in G.$

- $\mathcal{M}_c(G)$ is generally not a smooth manifold but there is a dense, open smooth locus $\mathcal{M}_c^\circ(G) \subset \mathcal{M}_c(G)$.

- We shall restrict our attention to the sublocus $\mathcal{M}^{\circ}_{c}(G,\vec{\Theta})$ of flat connections whose holonomy around puncture p_i is contained within the conjugacy class Θ_i.



$$\longmapsto \quad \rho(\gamma)\in\Theta_i \quad \forall\, \rho\in\mathcal{M}^{\circ}_{c}(G,\vec{\Theta})$$

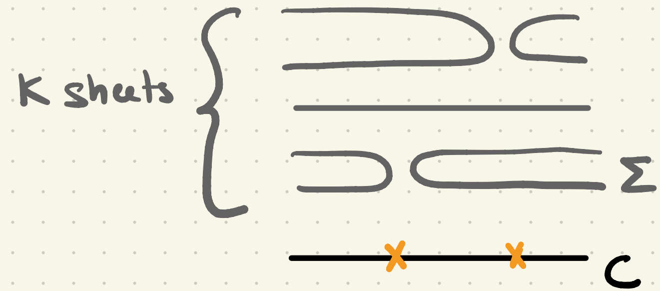
Recall: a conjugacy class for G is the orbit of an element $g\in G$ under conjugation,

$$\Theta(g) := \{\, hgh^{-1} : h\in G \,\}.$$

1) Spectral networks

Let $\iota : \Sigma \hookrightarrow T^*C$ be a branched K-fold cover,

$$\pi : \Sigma \xrightarrow{K:1} C .$$



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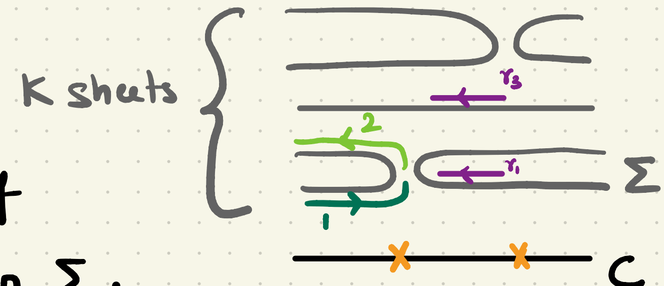
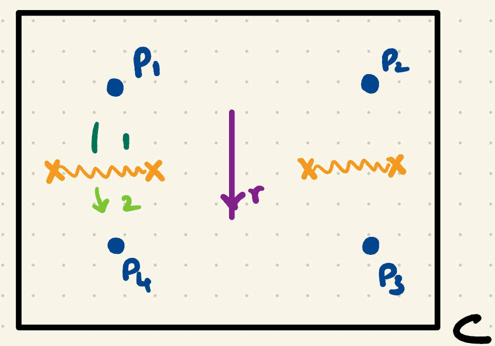


For convenience, fix a system of branch cuts trivialising the cover Σ.

For a curve $\gamma \subset C$, we denote by γ_i the lift to sheet i of Σ.



Fix a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

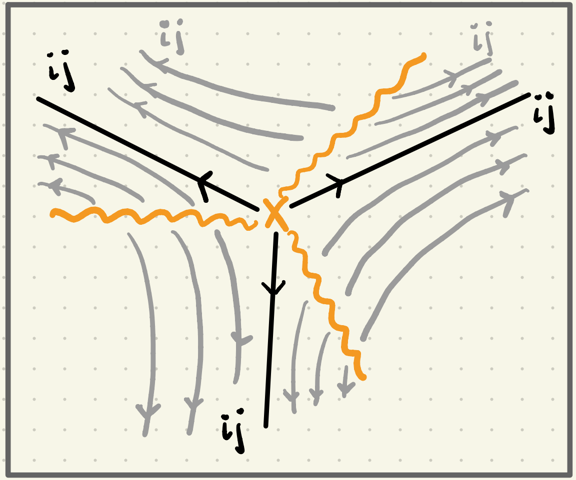
dfn. A curve $\gamma \subset C$ is said to be an ij-trajectory if

$$e^{-i\theta} \int_{\gamma_{ij}} \iota^* \lambda \in \mathbb{R}_{\geq 0} \quad \text{where } \gamma_{ij} := \gamma_i - \gamma_j$$

and $\lambda \in \Gamma(T^*C, T^*T^*C)$ is the Liouville one-form.

(in local coordinates, $\lambda = y\,dx$)

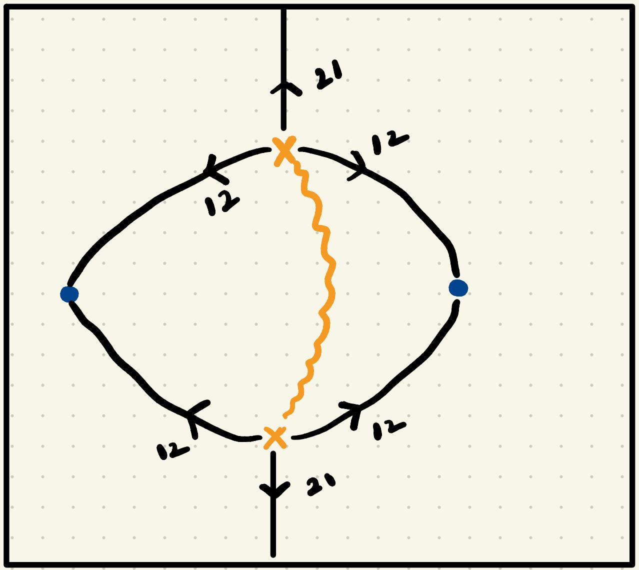
(Related to Morse theory, exact WKB)



An ij-trajectory is said to be critical if it terminates on a branch point. We orient all critical trajectories away from branch points.

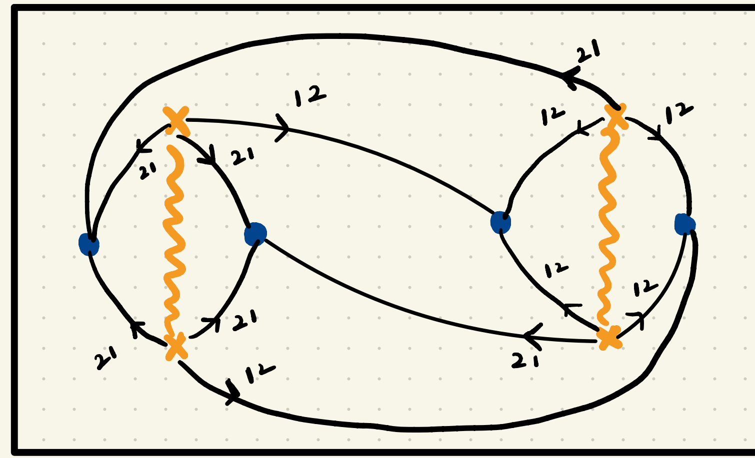
The collection of all branch points, critical trajectories, labels, and orientations is called a spectral network $\mathcal{W}(\Sigma, \theta)$.

Some simple examples:



$C = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$K = 2$



$C = \mathbb{P}^1 \setminus \{-1, 0, 1, 2\}$

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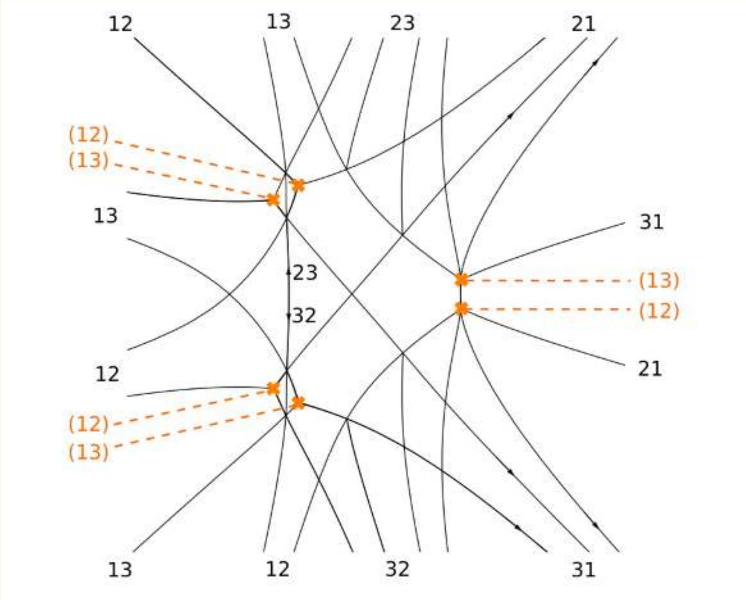


[HN20]

$C = \mathbb{P}^1 \setminus \{\infty\}$

$K = 3$

2) Abelianisation

Fix a rank K vector bundle $E \to C$. Then,

$$\mathcal{M}_C(GL(E)) \simeq \mathcal{M}_C(GL(K)).$$

Likewise, for a line bundle $L \to \Sigma$,

$$\mathcal{M}_\Sigma(GL(L)) \simeq \mathcal{M}_\Sigma(\mathbb{C}^\times).$$

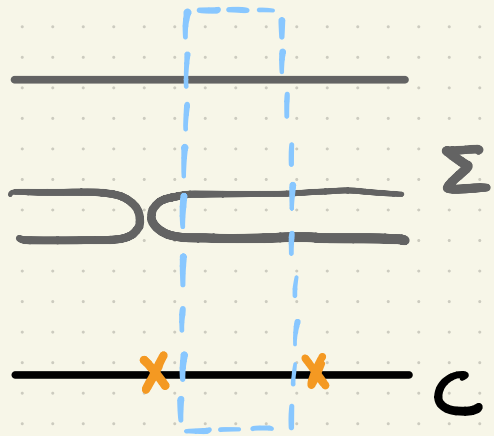
For a choice of a spectral network $\mathcal{W}$, we dfn. a

$\mathcal{W}$-abelianisation map $\quad \Phi_{\mathcal{W}} : \mathcal{M}^{\blacksquare}_C(GL(K)) \xrightarrow{\sim} \mathcal{M}^{\blacksquare}_\Sigma(\mathbb{C}^\times).$

Sketch of construction:

- Every connected component $\tilde{C}$ in $C \setminus W$ is contractible.

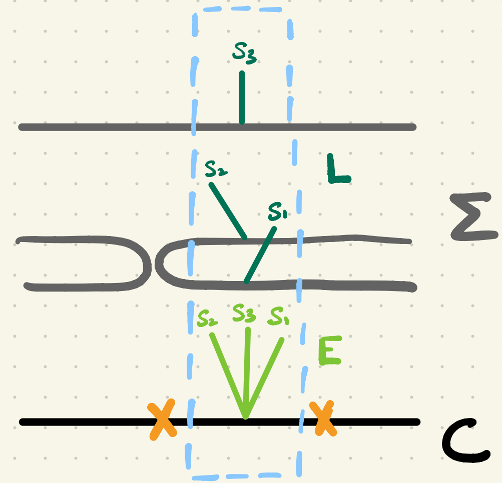
  Also, $\pi^{-1}(\tilde{C}) \cong \tilde{C} \sqcup \cdots \sqcup \tilde{C}$ (K copies).



- We may find a local eigenbasis $\{s_i\}$ for $E|_{\tilde{c}}$ for the $GL(k)$- connection, $\nabla s_i = d_i \otimes s_i$,

  for $i = 1, \ldots k$ and $\nabla \in \mathcal{M}_c(GL(k))$.

  We interpret s_i as being the generator of $L|_{\pi^{-1}(\tilde{c})}$ on the i^{th} sheet.



(Note that there is a freedom to rescale $s_i \mapsto \lambda s_i$)

- To consistently dfn. this on all of C, we require that across an ij-trajectory,

$$s_i \longmapsto s_i' = s_i + c_{ij} s_j,$$
$$s_k \longmapsto s_k' = s_k, \quad k \neq i.$$

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $K=4$, and a 13-trajectory

$$(s_1, \dots s_i, \dots, s_k) \xrightarrow{ij} (s_1, \dots \; s_i + c_{ij} s_j, \dots s_k)$$

- This ensures that $L|_{\pi^{-1}(\tilde{C})}$ extends to all of L.

3) Spectral coordinates

For $\nabla \in \mathcal{M}_C(GL(K))$, let $\nabla^{ab} := \Psi_W(\nabla)$.

For $\gamma \in H_1(\Sigma; \mathbb{Z})$, we find interesting nos.

$$\mathcal{X}_\gamma := \mathrm{hol}_\gamma \nabla^{ab} \in \mathbb{C}^\times.$$

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- They are gauge invariant!
- They are multiplicative fns. on $\mathcal{M}_C(GL(K))$, i.e.

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'} \quad \text{for } \gamma, \gamma' \in H_1(\Sigma; \mathbb{Z}).$$

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$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'} \text{ for } \gamma, \gamma' \in H_1(\Sigma; \mathbb{Z}).$$
- They also satisfy $\boxed{\quad\quad}$.

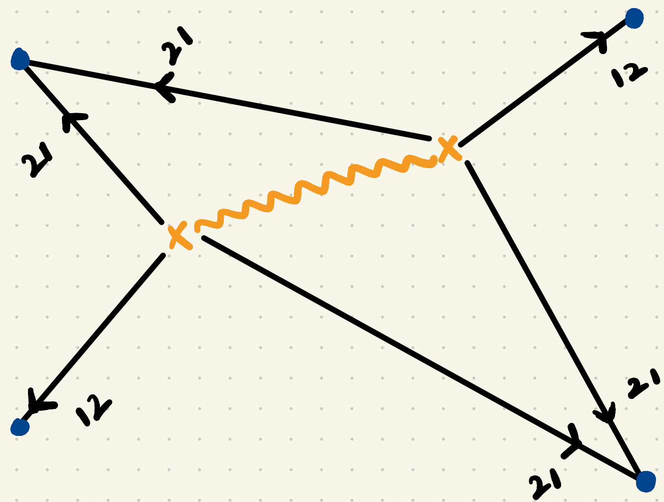
If we fix a set of generators $\{r_i\}$ for $H_1(\Sigma;\mathbb{Z})$, then the fns. $x_i := x_{r_i}$ dfn. coordinates on $\mathcal{M}_c(GL(K))$.

These are called spectral coordinates. They uniquely characterise $\nabla \in \mathcal{M}_c^{\blacksquare}(GL(K))$ in the local patch

$$\mathcal{M}_c^{\blacksquare}(GL(K)) \subset \mathcal{M}_c(GL(K)).$$

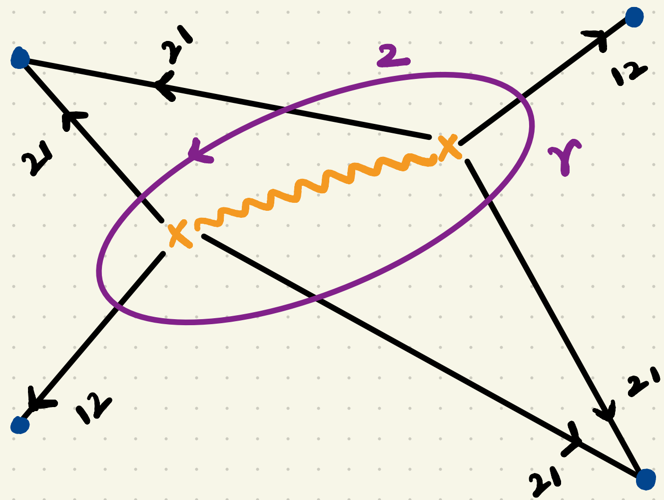
- Corollary: $\dim \mathcal{M}_c(GL(K)) = \dim H_1(\Sigma;\mathbb{Z})^{*}$

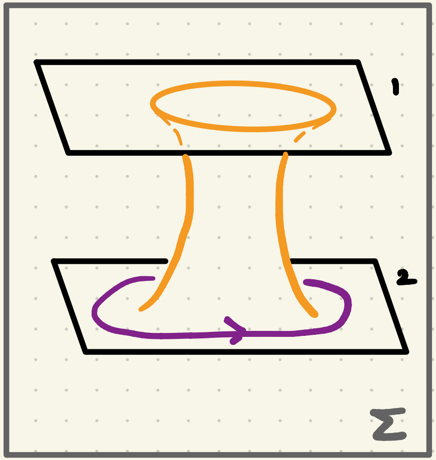
Example: For $K=2$, and the network shown below.



Example: For $K=2$, and the network shown below.

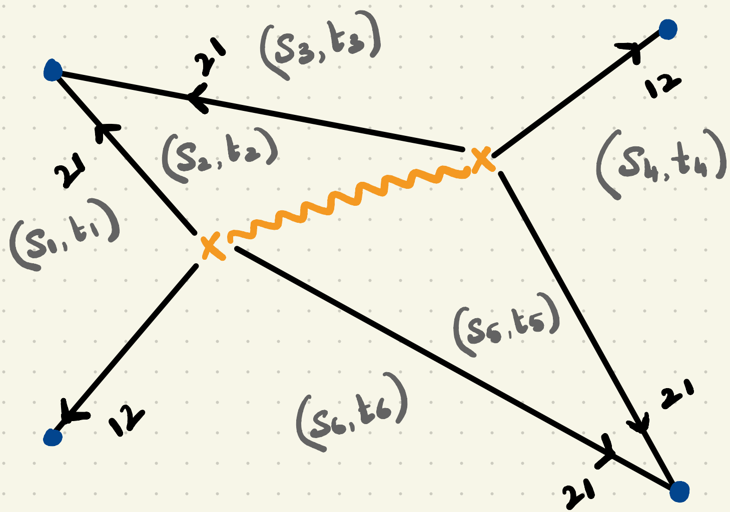
We will compute $\mathcal{X}_\gamma$ for γ as below.





Example: For $K=2$, and the network shown below.

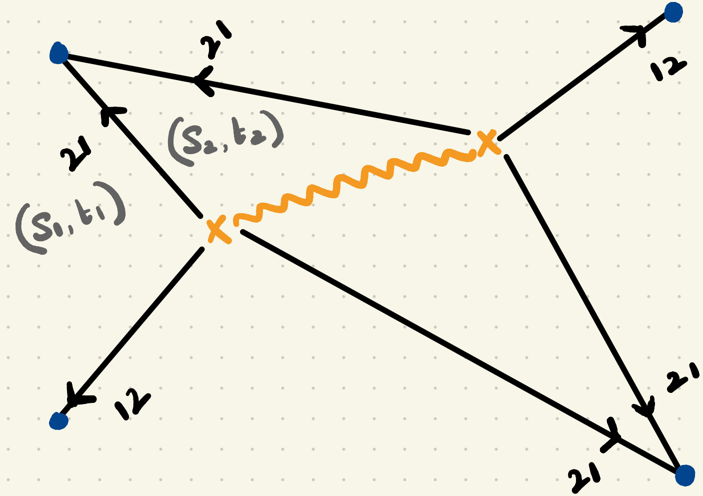
We pick an eigenbasis $\{s,t\}$ in each patch.



Crossing the 21- trajectory from region 1 to 2,

$$s_1 \longrightarrow s_1' = s_1.$$

$$t_1 \longrightarrow t_1' = t_1 + c s_1.$$



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$$s_1 \longrightarrow s_1' = s_1.$$

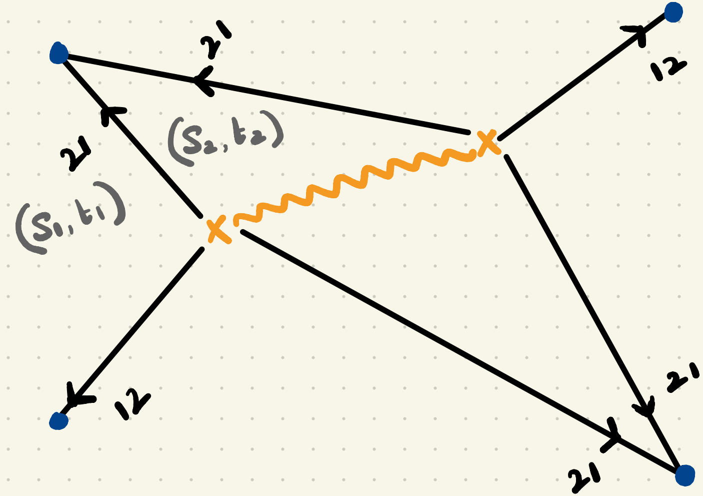
$$t_1 \longrightarrow t_1' = t_1 + c s_1.$$

Rescale $s_1 = s_2$.

$$\Rightarrow \quad t_1' = \lambda_{12} t_2 \, , \quad \lambda_{12} \neq 0.$$

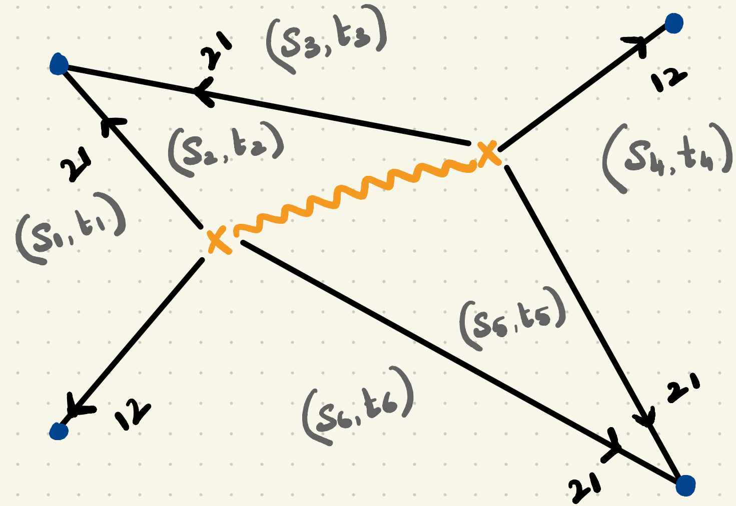
We evaluate

$$\lambda_{12} = \frac{t_1' \wedge s_1}{t_2 \wedge s_1} = \frac{t_1 \wedge s_1}{t_2 \wedge s_2}.$$



We continue this until we arrive back in region 1.

$$\lambda_{23} = \frac{t_2 \wedge s_2}{t_3 \wedge s_3}, \quad s_2 = s_3,$$



$$\lambda_{34} = 1, \quad t_3 = t_4,$$

$$\lambda_{45} = \frac{t_4 \wedge s_4}{t_5 \wedge s_5}, \quad s_4 = s_5,$$

$$\lambda_{56} = \frac{t_5 \wedge s_5}{t_6 \wedge s_6}, \quad s_5 = s_6, \text{ and}$$

$$\lambda_{61} = 1, \quad t_6 = t_1.$$

Going around γ, we see that

$$t_1 \longrightarrow \Lambda_\gamma t_1$$

where $\Lambda_\gamma = \lambda_{12}\lambda_{23}\lambda_{34}\lambda_{45}\lambda_{56}\lambda_{61}$

$$= \frac{t_1 \wedge s_1}{t_3 \wedge s_3} \frac{t_4 \wedge s_4}{t_6 \wedge s_6}. \qquad \left(\begin{matrix} s_1 = s_3, \ s_4 = s_6, \\ t_3 = t_4, \ t_1 = t_6 \end{matrix} \right)$$

By definition of holonomy, this is the spectral coordinate $\mathcal{X}_\gamma = P_\gamma$.

4) More applications

1. The inverse non-abelianisation map

$$\Psi_W^{-1} : \mathcal{M}_\Sigma(\mathbb{C}^\times) \longrightarrow \mathcal{M}_C(GL(K)).$$

2. Compute knot invariants

3. Compute global, analytic solutions to Schrödinger equations on arbitrary Riemann surfaces

4. Compute BPS invariants in certain supersymmetric theories.

Thank you for your attention!