Open Associativity Equations and ADE Singularities

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1 ISOLATED SURFACE SINGULARITIES

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w/ critical pt at 0

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germ of function

w/ critical pt at 0

The algebraic variety $V(f) := \{ z \in \mathbb{C}^n : f(z) = 0 \}$ is singular at 0.

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Simple surface singularities were classified by ARNOLD

A : $A_1, A_2, A_3, \dots$

D : $D_4, D_5, D_6, \dots$

E : E_6, E_7, E_8

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e.g. singularity of type D_ℓ:

$$f(z,w) = z^{\ell-1} + zw^2.$$

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A **DEFORMATION / UNFOLDING** is a germ

$$F : (\mathbb{C}^2 \times \mathbb{C}^e, 0) \longrightarrow (\mathbb{C}, 0)$$

w/ $F|_{\mathbb{C}^2 \times \{0\}} = f$.

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PARAMETER SPACE

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We denote $(z, w, \underline{a}) \in \mathbb{C}^2 \times \mathbb{C}^\ell$, $\quad F_{\underline{a}} := F|_{\mathbb{C}^2 \times \{\underline{a}\}}$.

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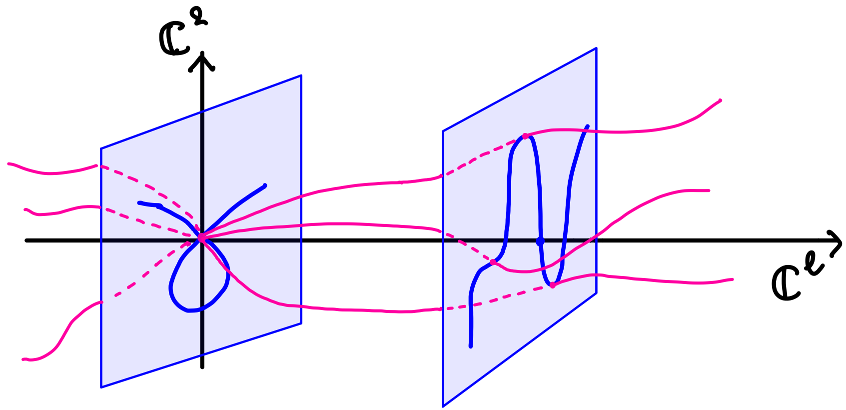

CRITICAL SPACE

$$Cr(F) := V(J(F))$$

$$J(F) = \left\langle \frac{\partial F}{\partial z}, \frac{\partial F}{\partial w} \right\rangle$$

JACOBIAN IDEAL

1 ISOLATED SURFACE SINGULARITIES

KODAIRA - SPENCER MAP

$$T\mathbb{C}^{\ell} \longrightarrow \mathbb{C}[z,w,\underline{a}]/J(F)$$

$$\partial/\partial a_\mu \longmapsto \frac{\partial F}{\partial a_\mu} + J(F)$$

JACOBIAN RING

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THM F MINIVERSAL $\Longleftrightarrow$ $T_0\mathbb{C}^{\ell} \xrightarrow{\sim} \mathbb{C}[z,w] \Big/ J(f)$

1 ISOLATED SURFACE SINGULARITIES

$\boxed{\text{THM}}$ F MINIVERSAL $\iff$ $T_0 \mathbb{C}^\ell \xrightarrow{\sim} \mathbb{C}[z,w]/J(f)$

SO: for a given basis $\phi_1, \dots, \phi_\ell$ of $\mathbb{C}[z,w]/J(f)$

$$F(z,w,\underline{a}) = f(z,w) + a_1\phi_1(z,w) + \cdots + a_\ell\phi_\ell(z,w)$$

is a miniversal deformation.

Notice that $\phi_\mu = \dfrac{\partial F}{\partial a_\mu}$.

1 ISOLATED SURFACE SINGULARITIES

e.g. type D_ℓ :

$$f(z,w) = z^{\ell-1} + zw^2,$$

$\mathbb{C}[z,w]/\langle f_z, f_w\rangle$ admits basis $\{z^{\ell-2}, \dots, z, 1, w\}$.

Hence, a miniversal deformation is given by:

$$F(z,w,\underline{a}) = z^{\ell-1} + zw^2 + a_1 z^{\ell-2} + \dots + a_{\ell-1} + a_\ell w.$$

1 ISOLATED SURFACE SINGULARITIES

RMK If F is miniversal, the Kodaira-Spencer map gives an isomorphism of v.s. at each pt in $\mathbb{C}^\ell$:

$$T_{\underline{a}}\mathbb{C}^\ell \xrightarrow{\sim} \mathbb{C}[z,w] \Big/ J(\underline{F_{\underline{a}}})$$

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IDEA ENDOW $T\mathbb{C}^\ell$ W/ A MULTIPLICATION by requiring that these maps be isomorphisms of rings.

1 ISOLATED SURFACE SINGULARITIES

FROBENIUS MANIFOLD

That is, a complex manifold M equipped w/

1. a commutative, associative product $\circ : \mathcal{X}_M \otimes_{\mathcal{O}_M} \mathcal{X}_M \longrightarrow \mathcal{X}_M$.

2. a symm., non-deg. bilinear form $\eta : \mathcal{X}_M \otimes_{\mathcal{O}_M} \mathcal{X}_M \longrightarrow \mathcal{O}_M$.

3. two distinguished holomorphic V.F. $e, E \in \mathcal{X}_M(M)$.

satisfying compatibility conditions.

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A **FROBENIUS MANIFOLD** a complex manifold M equipped w/

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satisfying compatibility conditions, including flatness and $\circ$-compatibility of η, $\eta(x \circ y, z) = \eta(y, x \circ z)$

e being the IDENTITY of $\circ$.

1 ISOLATED SURFACE SINGULARITIES

THM (Saito) The bilinear form on $T\mathbb{C}^{\ell}$:

$$\eta\left(\frac{\partial}{\partial a_\mu}, \frac{\partial}{\partial a_\nu}\right) := \sum_{x \in Cr(F)} \operatorname{Res}_x \left\{ \frac{\partial F}{\partial a_\mu} \frac{\partial F}{\partial a_\nu} \frac{dz \wedge dw}{F_z \, F_w} \right\}$$

is flat, non-degenerate and compatible with $\circ$.

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GROTHENDIECK'S RESIDUES

1 ISOLATED SURFACE SINGULARITIES

E (EULER V.F.) gives "homogeneity" of the structure.

For ADE singularities:

$\exists!$ grading of $\mathbb{C}[z,w]$ making f homogeneous of degree h.

A_ℓ : $h = \ell+1$,
D_ℓ : $h = 2(\ell-1)$,
E_6 : $h = 12$,
E_7 : $h = 18$,
E_8 : $h = 30$.

① ISOLATED SURFACE SINGULARITIES

E (EULER V.F.) gives "homogeneity" of the structure.

For ADE singularities:

$\exists!$ grading of $\mathbb{C}[z,w]$ making f homogeneous of degree h.

This extends uniquely to a grading of $\mathbb{C}[z,w,\underline{a}]$ making the deformation homogeneous of degree h.

$$E := \frac{1}{h} \sum_{\mu=1}^{\ell} (\deg a_\mu)\, a_\mu \frac{\partial}{\partial a_\mu}$$

1 ISOLATED SURFACE SINGULARITIES

e.g. for a singularity of type D_ℓ:

$f(z,w) = z^{\ell-1} + zw^2 \in \mathbb{C}[z,w]$, $h = 2(\ell-1)$.

$\Rightarrow$ $\deg z = 2$, $\deg w = 2(\ell-2)$.

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e.g. for a singularity of type D_ℓ:

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$\Longrightarrow$ $\deg z = 2$, $\deg w = \ell - 2$.

$$F(z,w) = z^{\ell-1} + zw^2 + a_1 z^{\ell-2} + \cdots + a_{\ell-1} + a_\ell w \in \mathbb{C}[z,w,\underline{a}]$$

$\deg a_1 = 2$, $\deg a_2 = 4$, $\ldots$, $\deg a_{\ell-1} = 2(\ell-1)$, $\deg a_\ell = \ell$.

$$E = \frac{1}{\ell-1} \sum_{\mu=1}^{\ell-1} \mu\, a_\mu \frac{\partial}{\partial a_\mu} + \frac{\ell}{2(\ell-1)} a_\ell \frac{\partial}{\partial a_\ell}.$$

1 ISOLATED SURFACE SINGULARITIES

Associated to any Frobenius manifold there is a solution Φ to the WITTEN-DIJKGRAAF-VERLINDE-VERLINDE EQUATIONS.
(WDVV)

PREPOTENTIAL
or
FREE ENERGY

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In a system of FLAT COORDINATES $t_1, \dots, t_\ell$ for η

$$\eta\left(\frac{\partial}{\partial t_\mu}, \frac{\partial}{\partial t_\nu}\right) \in \mathbb{C}$$

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In a system of FLAT COORDINATES $t_1, \dots, t_\ell$ for η:

$$\frac{\partial}{\partial t_\mu} \circ \frac{\partial}{\partial t_\nu} = \sum_{\alpha=1}^{\ell} c^{\alpha}_{\mu\nu} \frac{\partial}{\partial t_\alpha} \longrightarrow \frac{\partial^3 \Phi}{\partial t_\mu \partial t_\nu \partial t_\rho} := \sum_{\alpha=1}^{\ell} \eta_{\rho\alpha} c^{\alpha}_{\mu\nu}$$

$$\boxed{\circ \text{ ASSOCIATIVE} \iff \Phi \text{ SOLVES WDVV EQUATIONS}}$$

1 ISOLATED SURFACE SINGULARITIES

$\boxed{\circ \text{ ASSOCIATIVE} \iff \Phi \text{ SOLVES WDVV EQUATIONS}}$

Homogeneity of the structure $\iff$ Φ is homogeneous.

FOR ADE SINGULARITIES:

One can find flat coordinates so that $\mathbb{C}[\underline{t}] \cong \mathbb{C}[\underline{a}]$ and $\deg t_\mu = \deg a_\mu$

$\Phi(\underline{t}) \in \mathbb{C}[\underline{t}]$ homogeneous of degree $2(h+1)$.

2 LANDAU-GINZBURG MODELS

Constructions of Frobenius manifolds generalising ADE singularities.

An LG-MODEL is a pair (λ, ω)

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1. λ is a family of meromorphic functions on a Riemann surface of fixed genus g w/ prescribed pole structure.

$\boxed{\text{LG SUPERPOTENTIAL}}$

2 LANDAU-GINZBURG MODELS

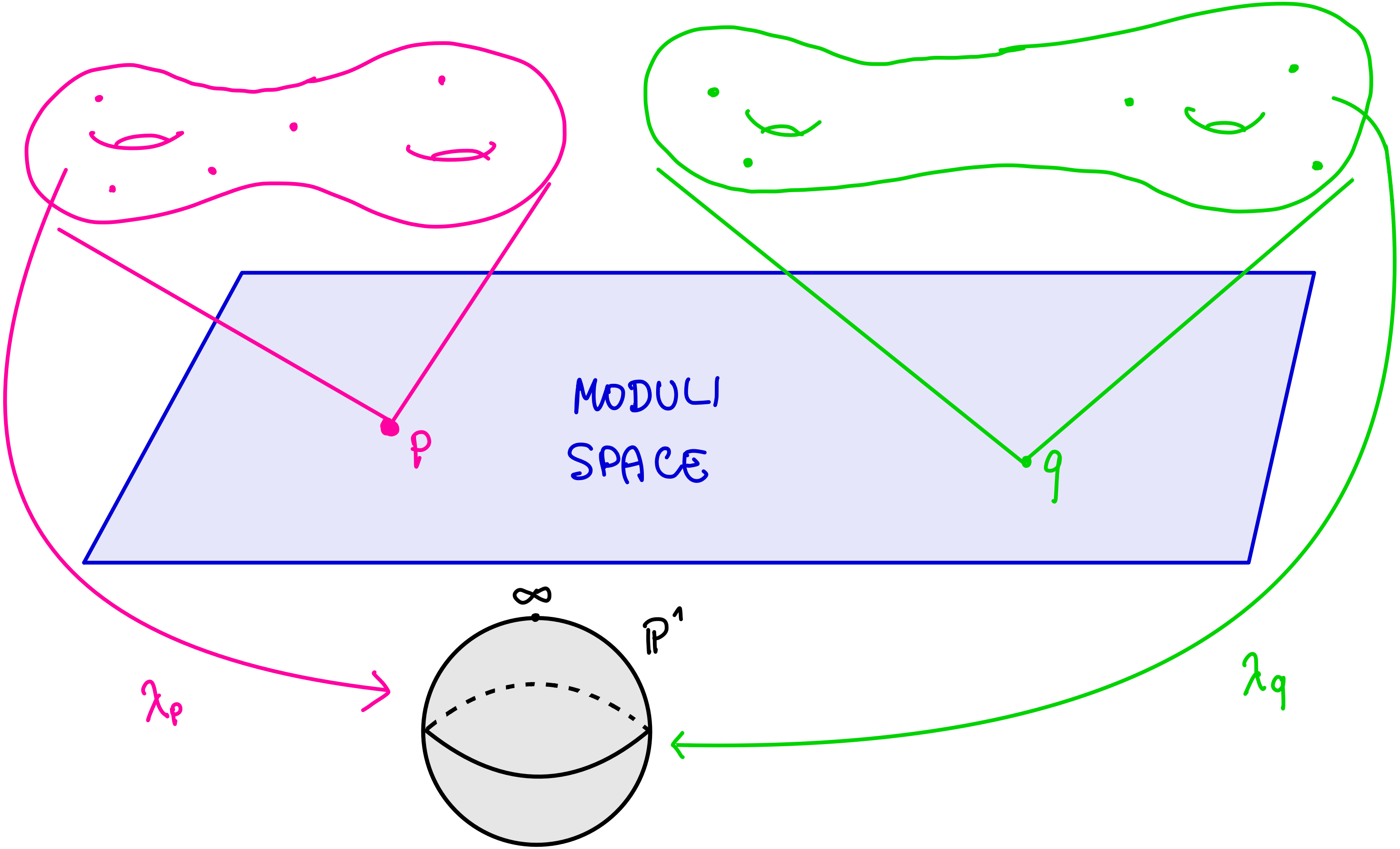
Constructions of Frobenius manifolds generalising Saito theory.

An LG-MODEL is a pair (λ, ω)

1. λ is a family of meromorphic functions on a Riemann surface of fixed genus g w/ prescribed pole structure.

2. ω is a family of meromorphic differentials on the same curve, satisfying some admissibility conditions.

PRIMARY DIFFERENTIAL/FORM


MODULI
SPACE
p
q
∞
$\mathbb{P}^1$
λ_p
λ_q

2 LANDAU-GINZBURG MODELS

The metric and multiplication are defined by:

$$\eta(X,Y) := \sum_{x \in Cr(\lambda)} \operatorname{Res}_x \left\{ X(\lambda)\, Y(\lambda) \frac{\omega^2}{d\lambda} \right\}.$$

$$c(X,Y,Z) := \sum_{x \in Cr(\lambda)} \operatorname{Res}_x \left\{ X(\lambda)\, Y(\lambda)\, Z(\lambda) \frac{\omega^2}{d\lambda} \right\}$$

$$\equiv \eta(X \circ Y, Z).$$

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e.g. LG MODEL for ADE singularities:

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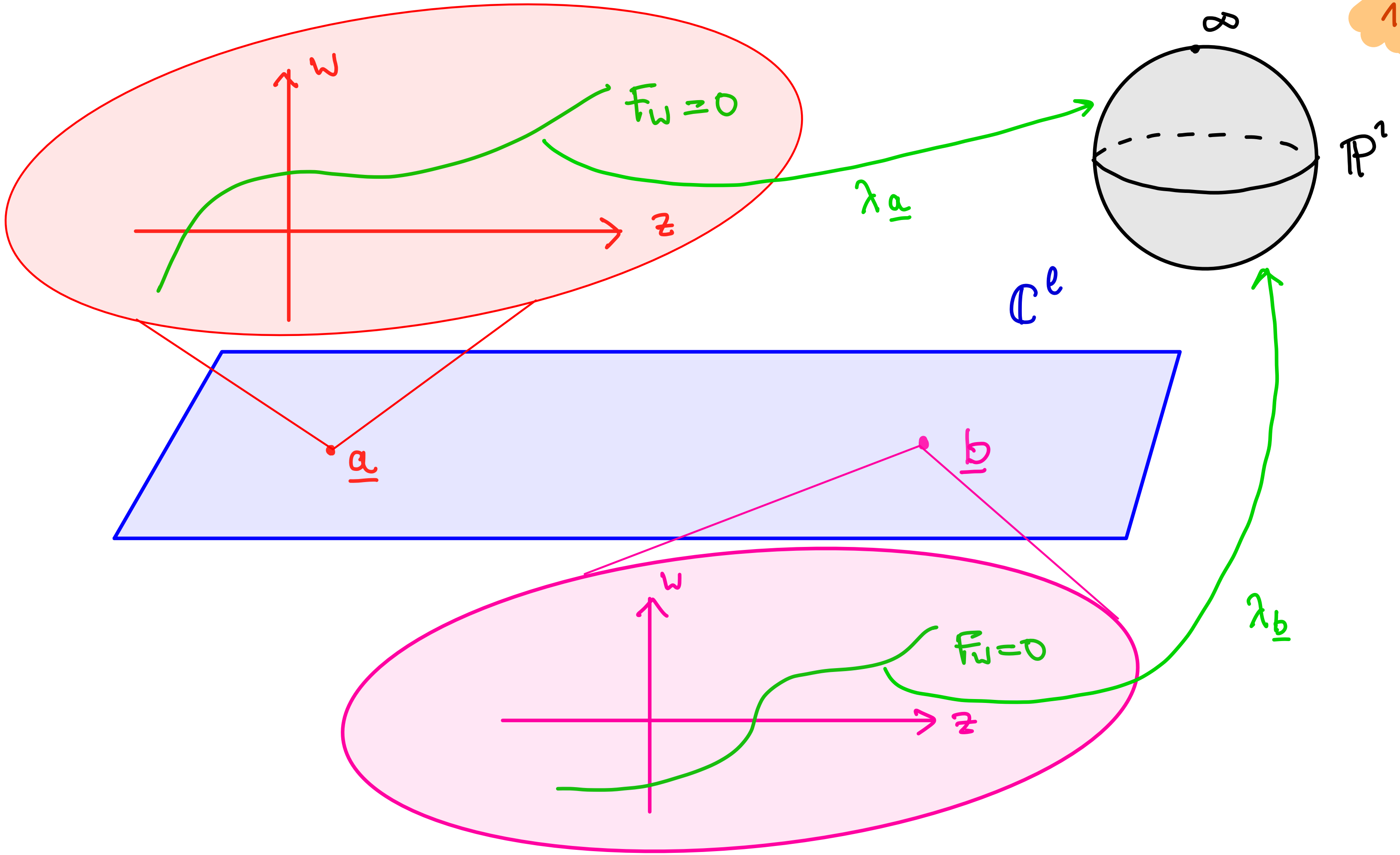
$$\boxed{C_F(\underline{a}) := \{ (z,w) \in \mathbb{C}^2 : \quad F_w(z,w,\underline{a}) = 0 \}}$$

This defines $C_F \hookrightarrow \mathbb{C}^2 \times \mathbb{C}^\ell$.

Let $\boxed{\lambda := F|_{C_F}}$ so that $\lambda_z = F_z$ on C_F.


w
z
$F_w = 0$
$\lambda_{\underline{a}}$
∞
$\mathbb{P}^1$
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$\underline{a}$
$\underline{b}$
w
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2 LANDAU-GINZBURG MODELS

e.g. SINGULARITY OF TYPE D:

$$F(z,w) = z^{\ell-1} + zw^2 + a_1 z^{\ell-2} + \cdots + a_{\ell-1} + a_\ell w.$$

$$F_w = 2zw + a_\ell$$

② LANDAU-GINZBURG MODELS

e.g. SINGULARITY OF TYPE D_ℓ:

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$$F_w = 2zw + a_\ell$$

$$\Rightarrow \quad \boxed{\lambda(z) = z^{\ell-1} + a_1 z^{\ell-2} + \dots + a_{\ell-1} - \frac{1}{4z} a_\ell^2}$$

FAMILY OF MEROMORPHIC FUNCTIONS ON $\mathbb{P}^1$
W/ POLES AT ∞ AND 0.

3 FLAT F-MANIFOLDS

That is a complex manifold M equipped w/

1) an associative, commutative product $\circ : \mathcal{X}_M \otimes_{\mathcal{O}_M} \mathcal{X}_M \longrightarrow \mathcal{X}_M$.
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A VECTOR (PRE) POTENTIAL $\Phi \in \mathcal{X}_M(\mathcal{U})$ can still be defined locally so that, in flat coordinates $t_1, \dots, t_n$ for ∇:

$$\frac{\partial}{\partial t_\mu} \circ \frac{\partial}{\partial t_\nu} = \sum_{\alpha=1}^{n} c^\alpha_{\mu\nu} \frac{\partial}{\partial t_\alpha} \longrightarrow \frac{\partial^2 \Phi_\alpha}{\partial t_\mu \partial t_\nu} := c^\alpha_{\mu\nu}$$

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$\boxed{\circ \text{ ASSOCIATIVE } \iff \Phi \text{ SOLVES ORIENTED WDVV EQUATIONS}}$

RMK If M is a Frobenius manifold with prepotential Φ, then it is an flat F-manifold with vector potential $\nabla \Phi$.

3 ALMOST-FLAT F-MANIFOLDS

An **EXTENSION** of a Frobenius manifold M is a flat F-manifold $\tilde{M}$ with a fibration $\pi : \tilde{M} \longrightarrow M$ such that:

1. $\pi_{*p} : T_p\tilde{M} \longrightarrow T_{\pi(p)}M$ is a ring-homomorphism.

2. $\tilde{\nabla}\big|_{\pi^*TM} = \pi^*\nabla$.

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$\boxed{\text{RMK}}$ $\operatorname{Ker}\pi_{*p}$ is an ideal in $T_p\tilde{M}$ $\forall p$.

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RMK $\operatorname{Ker}\pi_{*p}$ is an ideal in $T_p\tilde{M}$ $\forall p$.

RMK Flat sections of ∇ lift to flat sections of $\tilde{\nabla}$.
$\Rightarrow$ Flat coordinates $t_1, \dots, t_n$ on M can be completed to flat coordinates $t_1, \dots, t_n, x_1, \dots, x_r$ on $\tilde{M}$.

3 ALMOST-FLAT F-MANIFOLDS

The vector potential Φ on $\widetilde{M}$ can be chosen so that:

$$\boxed{\Phi = \nabla \underbrace{\Phi_0}_{\text{PREPOTENTIAL ON } M} + \Omega}$$

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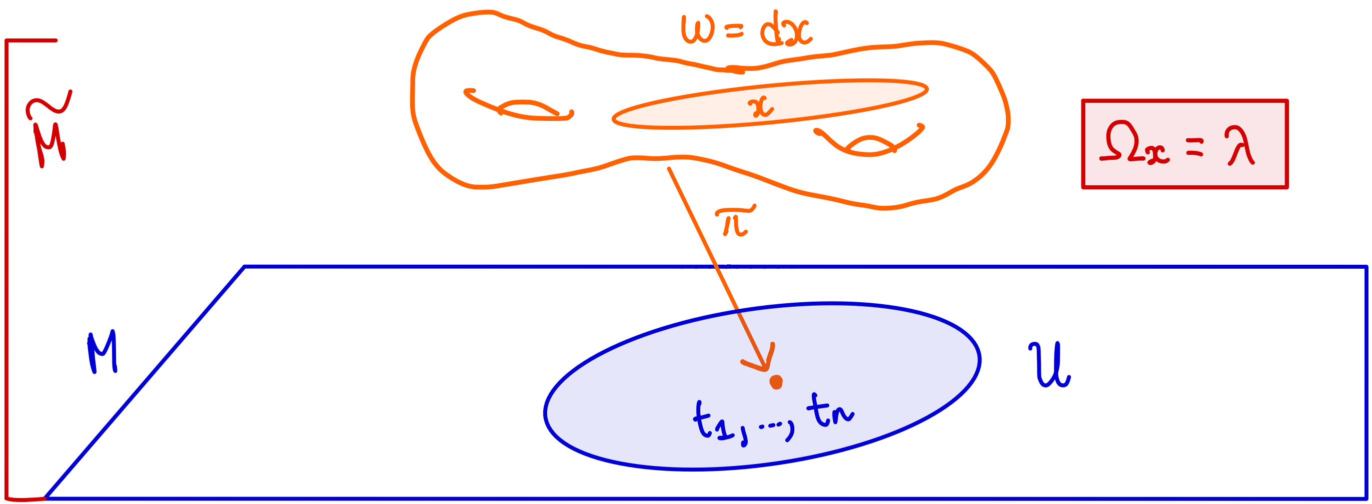
EXTENDED (PRE) POTENTIAL

$$\boxed{\tilde{\circ} \text{ ASSOCIATIVE} \iff \Omega \text{ SOLVES THE OPEN WDVV EQUATIONS}}$$

3 ALMOST-FLAT F-MANIFOLDS

(Almeida '25)

THM If a Frobenius manifold admits an LG-model, then it has a unique rank-one extension such that:



$$\boxed{\Omega_x = \lambda}$$

3 ALMOST-FLAT F-MANIFOLDS

e.g. singularity of type D_ℓ:

$$\lambda(x) = \frac{1}{2^{\ell-1}} x^{2(\ell-1)} + \frac{1}{2^{\ell-2}} a_1 x^{2(\ell-2)} + \dots + a_{\ell-1} - \frac{1}{2x^2} a_\ell^2 \quad , \qquad \omega = dx .$$

Given a system of flat coordinates $t_1, \dots, t_\ell$ for the ADE Frobenius manifold, $t_1, \dots, t_n, x$ is a system of flat coordinates for the extension and:

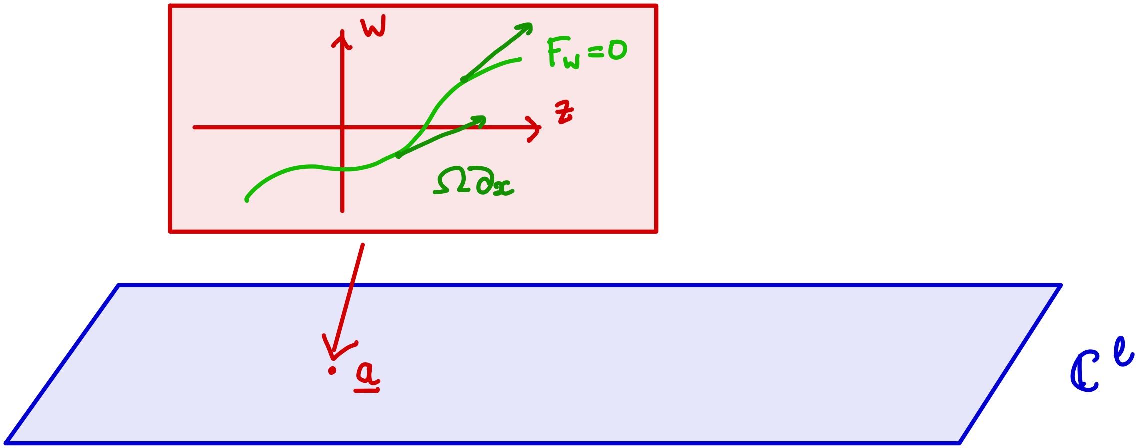
$$\boxed{\Omega(\underline{t}, x) = \frac{1}{2^{\ell-1}(2\ell-1)} x^{2\ell-1} + \frac{1}{2^{\ell-2}(2\ell-3)} a_1(\underline{t}) x^{2\ell-3} + \dots + a_{\ell-1}(\underline{t}) x + \frac{1}{2x} a_\ell(\underline{t})^2 + K(\underline{t})}$$

CONSTANT OF INTEGRATION computed to be zero by Basalaev and Buryak (2022).

4 RANK-TWO EXTENSIONS

Reason why we expect rank-two extensions in the ADE singularity cases:

the LG-model is a restriction of a polynomial to an algebraic curve in $\mathbb{C}^2$



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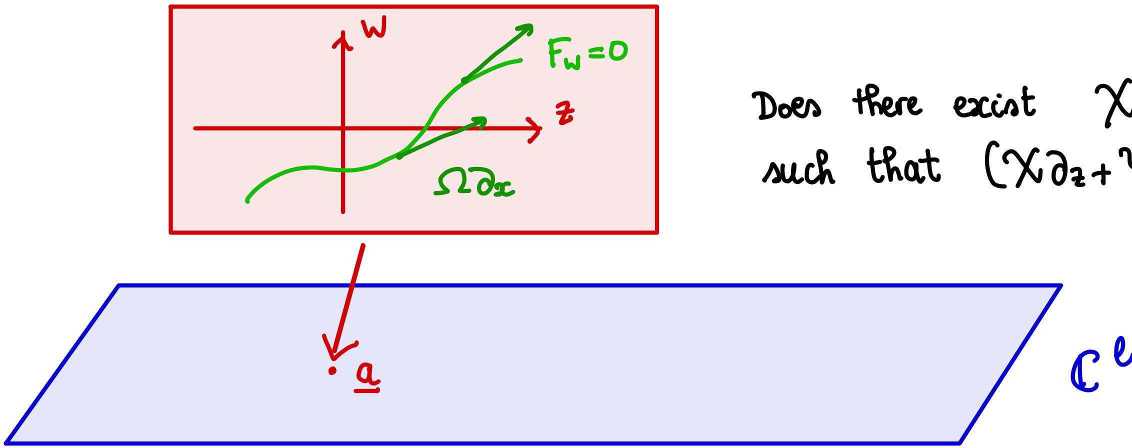

Does there exist $\chi\partial_z + \Psi\partial_w$ such that $(\chi\partial_z + \Psi\partial_w)|_{C_F} = \Omega\partial_x$?

4 RANK-TWO EXTENSIONS

CONJECTURE

The structures on the parameter spaces $\mathbb{C}^\ell$ of deformations of ADE singularities admit a unique POLYNOMIAL rank-two extension to $\mathbb{C}^2 \times \mathbb{C}^\ell$ which

a) is homogeneous w.r.t. the induced grading on $\mathbb{C}[z, w, \underline{a}]$.

b) restricts to the known rank-one extension on a suitable algebraic curve in $\mathbb{C}^2$.

Thank you

for

your attention !